A new look for the pion form factor

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TPP Sussex seminar – 12 May 2025

(based on 2410.13764 with Kubis, Reboud, van Dyk)

A simple model of form factors above the pair production threshold

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(based on 2410.13764 with Kubis, Reboud, van Dyk)

- Why are we interested in form factors?
- Overview of dispersive bounds
- What about above threshold data?
- Results
- Future outlook and summary

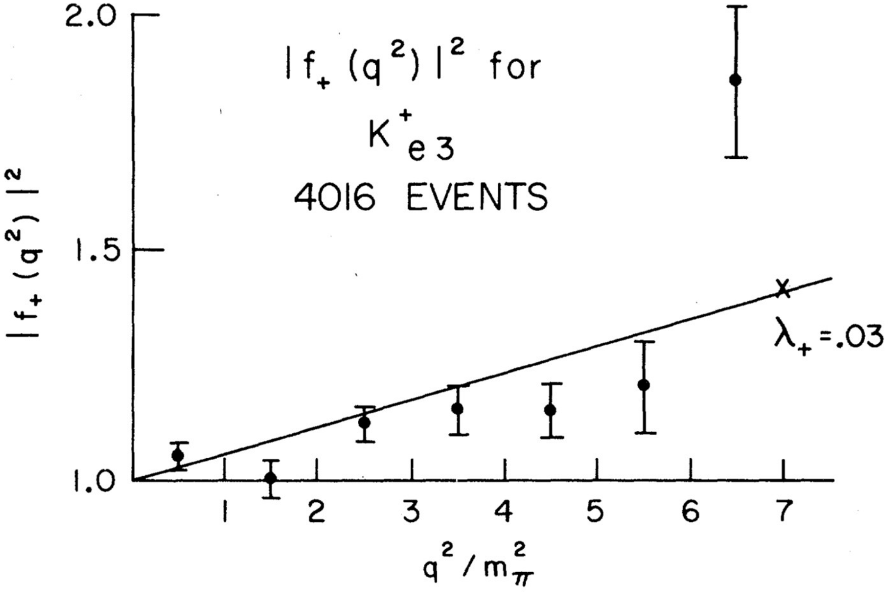
Why are we interested in form factors?

Why are we interested in form factors?

- Semi-leptonic decays are very interesting
  - E.g. for determining CKM elements, but also potential BSM
- Consider $K \to \pi$ which is used to extract V_{us}
- But $\tau \to K\pi\nu$ should also give access to V_{us}

Why are we interested in form factors?

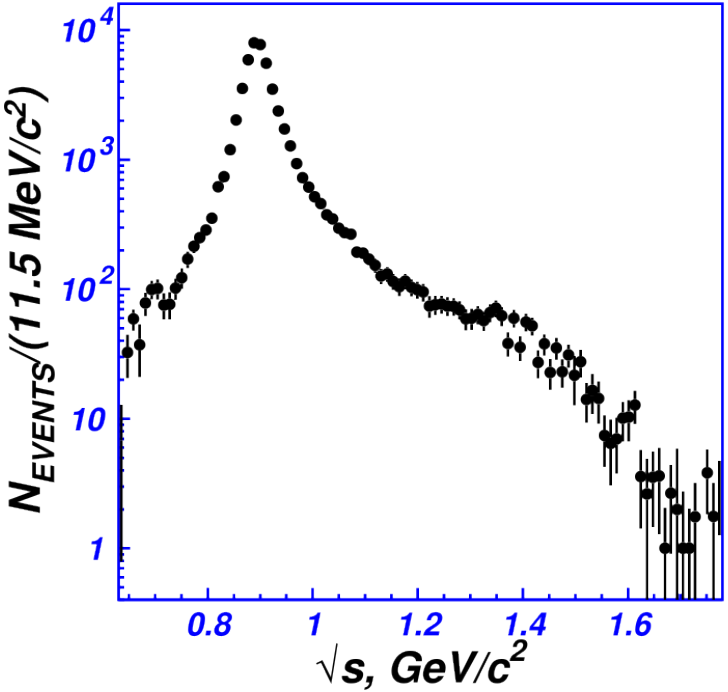
$K \to \pi e \nu$



Chiang, Rosen, Shapiro, Handler, Olsen, Pondrom 1972

$\tau \to K \pi \nu$

Belle 0706.2231



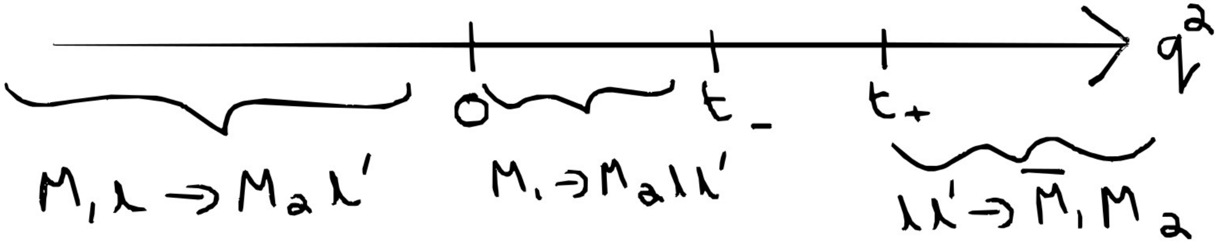
What is a form factor?

- Hadronic quantities
- $\langle M_2(p_2)|j|M_1(p_1)\rangle \sim F(q^2 = (p_1 - p_2)^2)$
  - $q^2 < 0 : M_1\ell \to M_2\ell'$
  - $0 < q^2 < t_- : M_1 \to M_2\ell\ell'$
  - $q^2 > t_+ : \ell\ell' \to \bar{M}_1 M_2$

$$t_\pm = (m_1 \pm m_2)^2$$

$$\xrightarrow{\text{crossing symmetry}} \langle \overline{M_1} M_2|j|0\rangle$$

What is a form factor?



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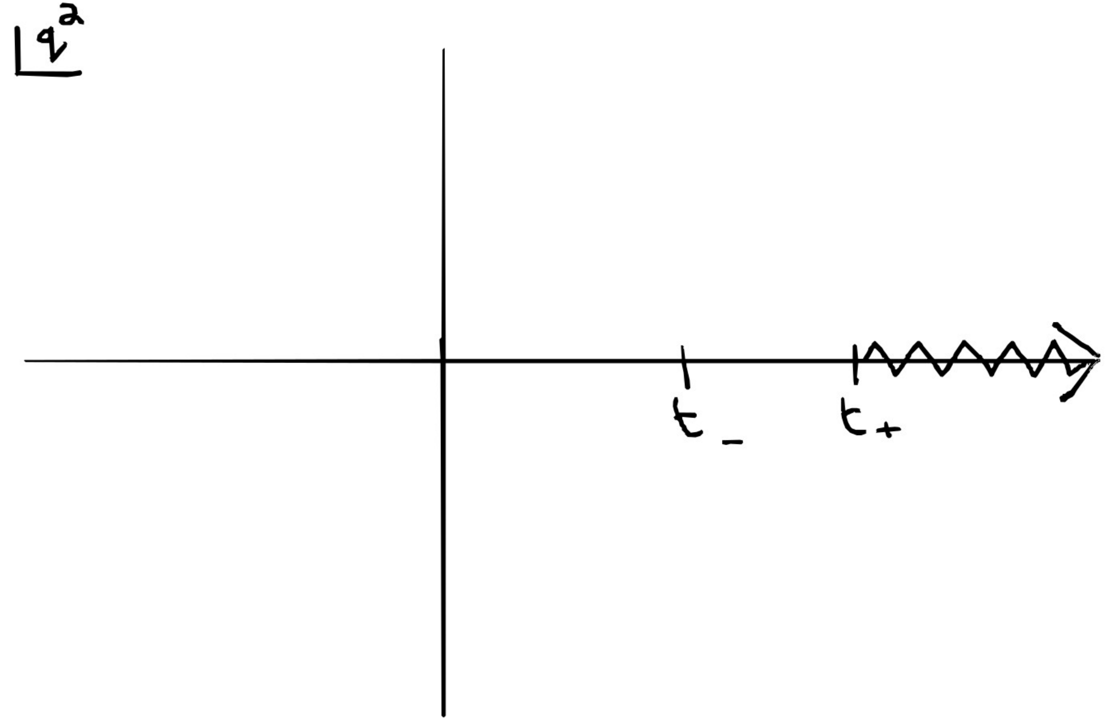
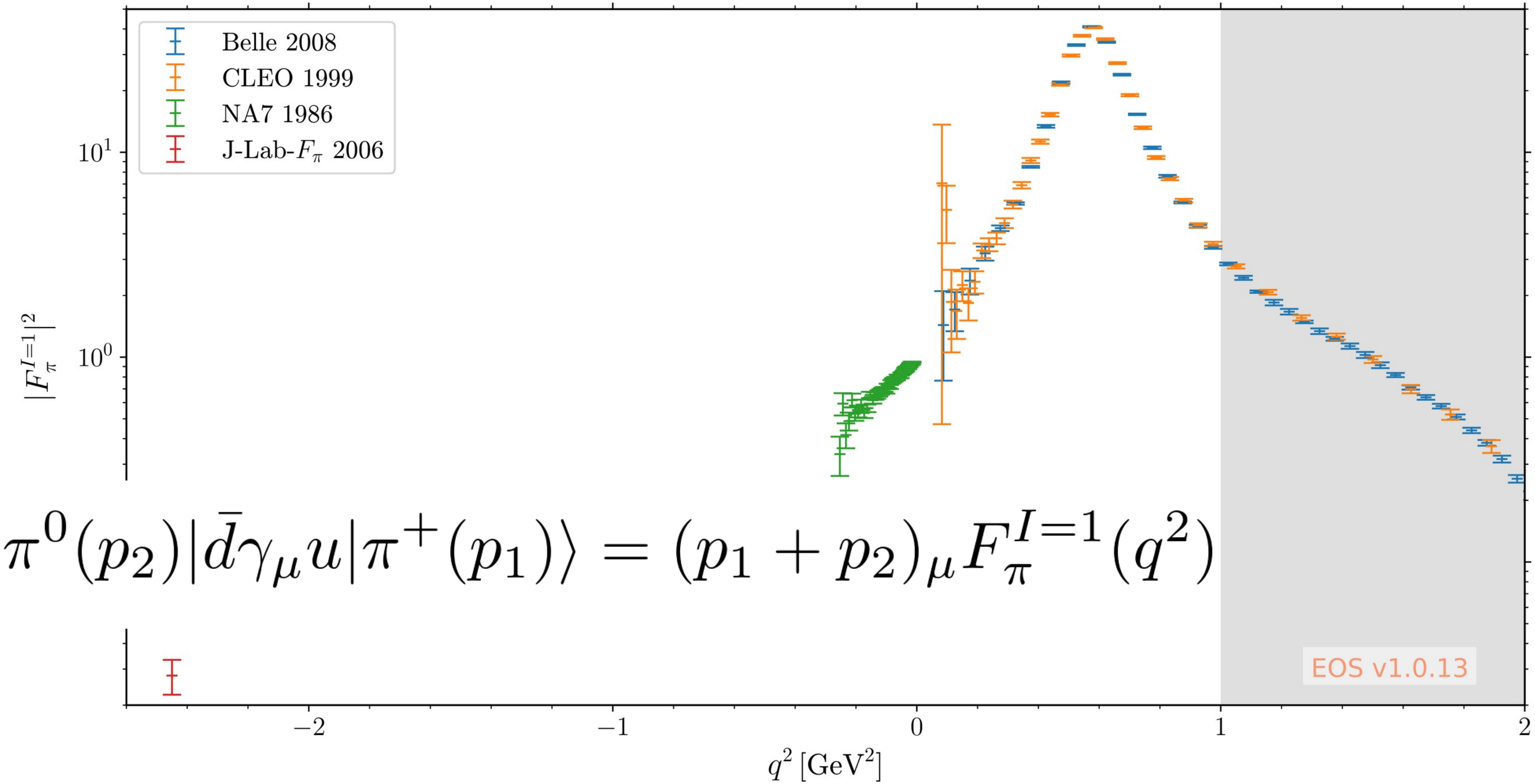


Simple case: pion form factor

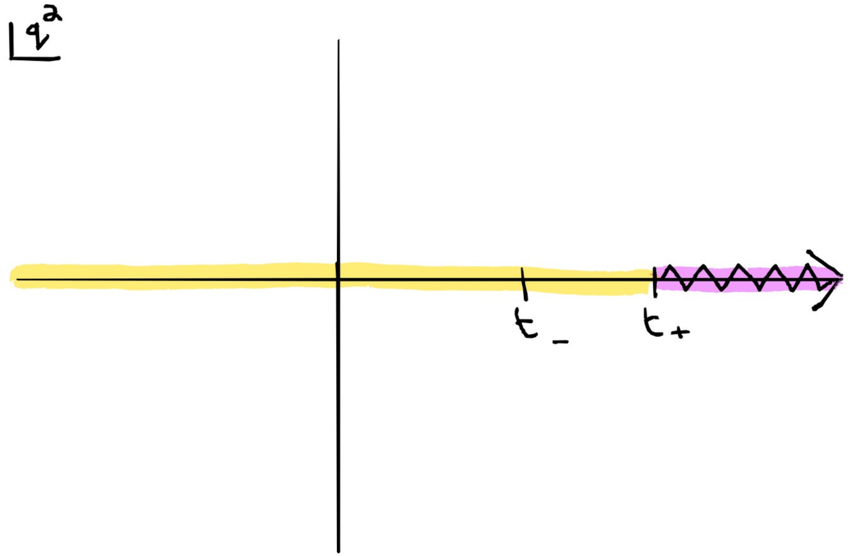


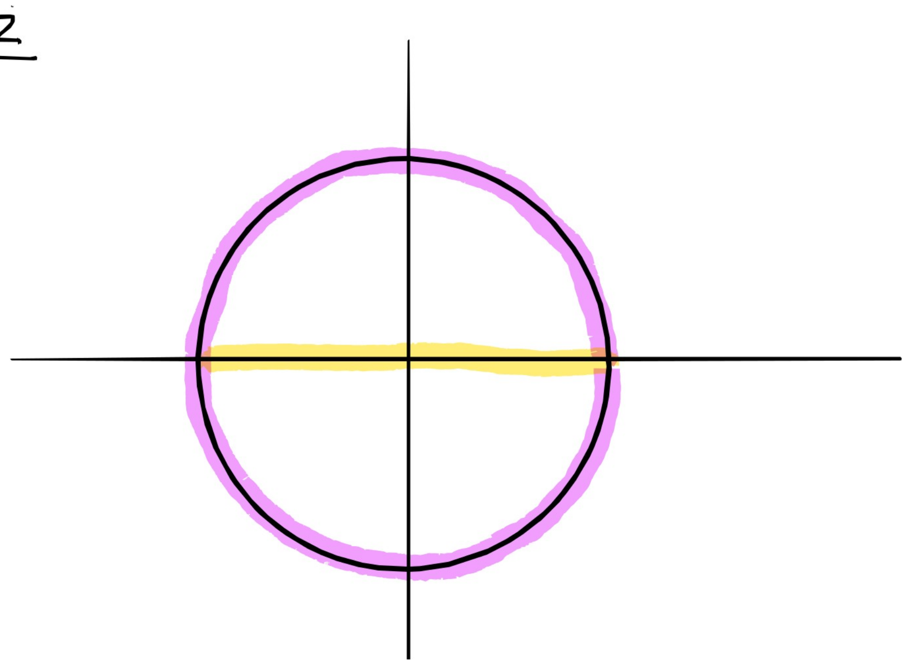
$$\frac{1}{\sqrt{2}}\langle\pi^0(p_2)|\bar{d}\gamma_\mu u|\pi^+(p_1)\rangle = (p_1+p_2)_\mu F_\pi^{I=1}(q^2)$$

How do we describe form factors?

- Common parameterisation uses a conformal mapping from q^2 plane to z
  - First used for form factors in Meiman (1963, JETP), Okubo (1971, PRD)
  - Made famous by BGL parameterisation

Conformal mapping





How do we describe form factors?

- Common parameterisation uses a conformal mapping from q^2 plane to z
- Why is this useful?
- Need to understand dispersive bounds…

Overview of dispersive bounds

Dispersive bounds in 1 slide

- Write $e^- \bar{\nu} \to \bar{u} d$ in both inclusive (i.e. perturbative quark level) and exclusive (sum over meson states) way
- Inclusive = Exclusive
  - Inclusive we calculate in QCD using OPE
  - Exclusive depends on form factor
  - We can drop terms to turn equality to inequality

Dispersive bounds in 3 slides

- Consider $\Pi(q^2) =$ ~~~O~~~
- Π is analytic, except on the positive real axis, where there are poles from resonances
- Use Cauchy to write $\Pi(q^2) = \oint dt \frac{\Pi(t)}{t-q^2}$
- Analytic structure means we can write this as $\Pi(q^2) = \int_{t_+}^{\infty} dt \frac{\mathrm{Im}\,\Pi(t)}{t-q^2}$

Dispersive bounds in 3 slides

- $\Pi(q^2) = \int_{t_+}^{\infty} dt \frac{\mathrm{Im}\,\Pi(t)}{t-q^2}$
- For q^2 very large and negative, LHS is calculable using an OPE
- While imaginary part related to on-shell intermediate states

Dispersive bounds in 3 slides

- $\Pi(q^2) = \int dt \frac{\mathrm{Im}\,\Pi}{t-q^2} \sim \int dt \frac{1}{t-q^2} \int_{\mathrm{P.S.}} \sum_X \langle 0|j|X\rangle\langle X|j^\dagger|0\rangle$
- Look just at two particle terms: $X = M_1 \bar{M}_2$
- By crossing symmetry, this is just our form factor!
- RHS is a sum of positive terms, so we can drop terms and just replace the equality with an inequality. This is the basic dispersive bound!

Simplifying the dispersive bound

- Often we write $F \sim \sum_i \alpha_i f_i$, for some functions f_i
- Ideally chosen such that, given our Cauchy integral, the RHS reduces to $\sum_i |\alpha_i|^2$
- So the dispersive bound reduces to a bound on parameters α_i

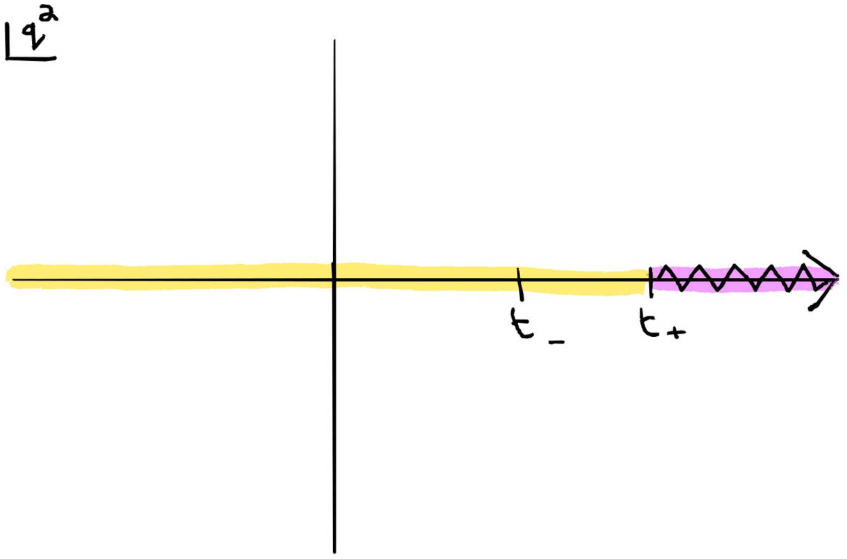
Simplifying the dispersive bound

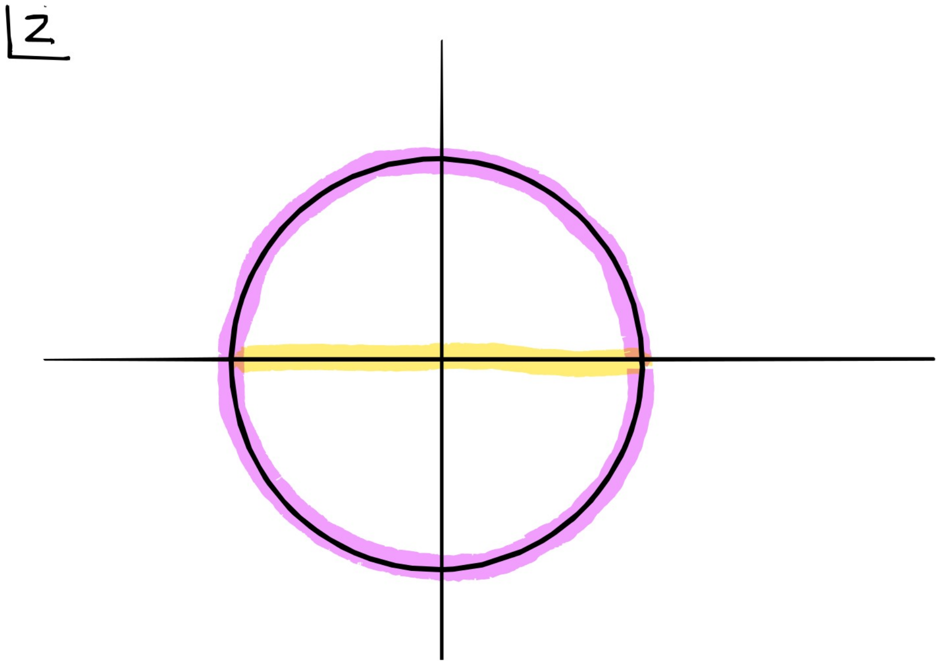
- Can be tricky to choose f_i properly
- For our analysis of the pion form factor, we did not find a nice choice
  - I will explain later what the issues are

Apply the conformal mapping

- Found $\int_{t_+}^{\infty} dt \frac{1}{\Pi(q^2)} \frac{1}{t-q^2} \int_{\mathrm{P.S.}} |F|^2 \leq 1$
- Everything in front of $|F|^2$ (inclusive calculation, Cauchy denominator, phase space factors) is grouped into an "outer function", written as $|\phi|^2$
  - Note the outer function is fixed for any transition
- Now change from q^2 to z

Apply the conformal mapping



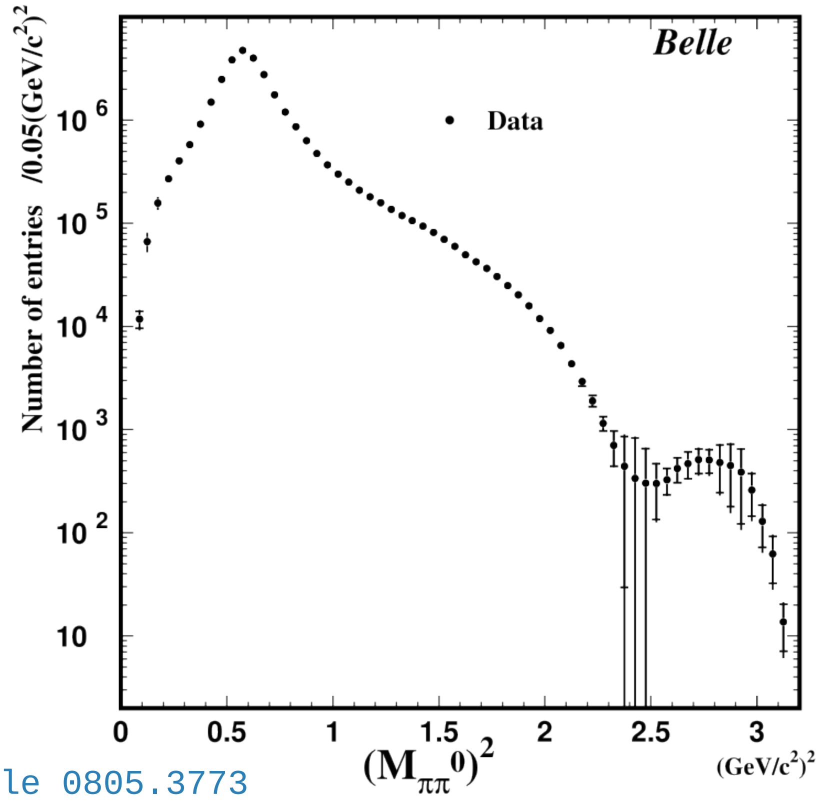


Apply the conformal mapping

- $\int_{t_+}^{\infty} \to \oint_{|z|=1}$
- Write $F = \frac{1}{\phi} \sum_i \alpha_i z^i$
- Polynomials z^i useful since $\oint_{|z=1|} z^i \bar{z}^j dz = \delta_{ij}$
- Dispersive bound become extremely simple!
- $$\underbrace{\int_{t_+}^{\infty} dt \, \frac{1}{\Pi(q^2)} \frac{1}{t-q^2} \int_{\mathrm{P.S.}}}_{|\phi|^2} |F|^2 \leq 1 \Rightarrow \sum_i |\alpha_i|^2 \leq 1$$

What about above threshold data?

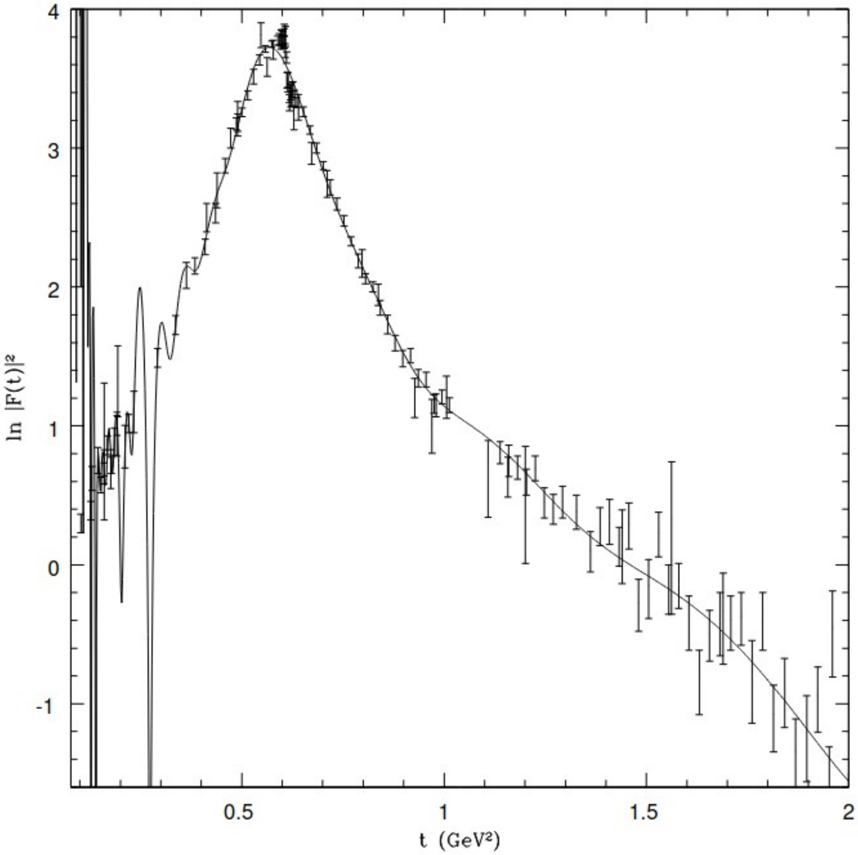
Data in the above threshold region



Belle 0805.3773

- Can we also use data here as part of the fit?
- In 1998, Buck & Lebed studied this problem

Data in the above threshold region



Buck, Lebed (hep-ph/9802369)

- Can we also use data here as part of the fit?
- In 1998, Buck & Lebed studied this problem
- They found no, get spurious oscillations near threshold

What went wrong?

- For $q^2 > t_+$, our expansion parameter has $|z| = 1$!
- Does the sum even converge?
- Yes – see section IV of Buck Lebed 1998
  - Roughly speaking: the form factor has a physically well defined quantity along the cut in q^2, Abel's theorem guarantees the series converges to that limit

Buck Lebed 1998

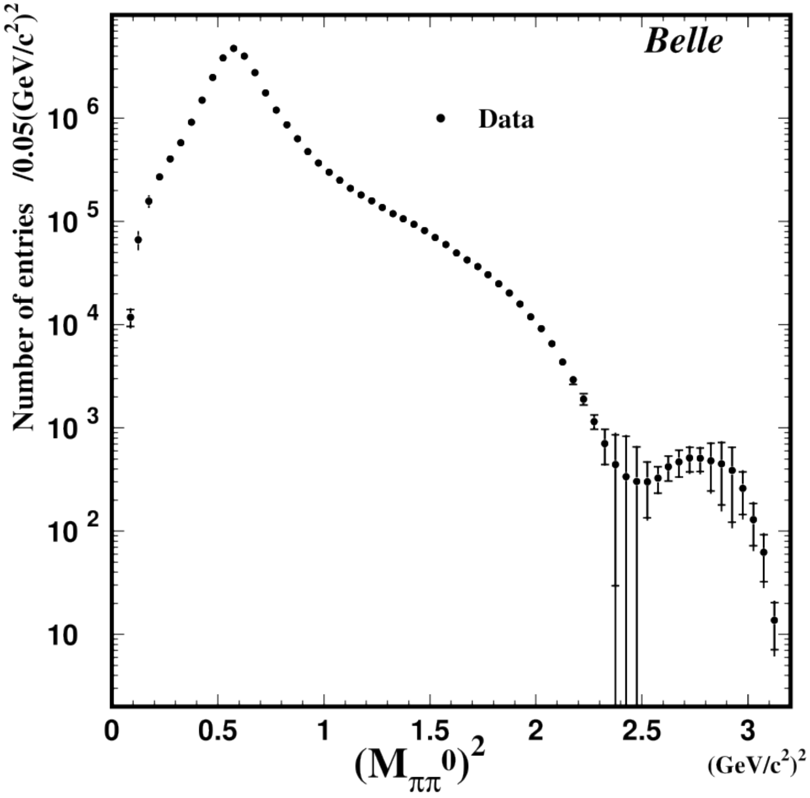
- An issue they discuss is that with $F = \frac{1}{\phi} \sum_i \alpha_i z^i$, F picks up two incorrect behaviours from ϕ
- ϕ has a zero at $q^2 = t_+$ => F blows up
- Asymptotic behaviour of ϕ as $q^2 \to \infty$ leads to $F(q^2 \to \infty) \sim (q^2)^{1/4}$

What’s wrong? And how do we fix it?

- Neither is physical
  - Experiment tells us F is finite near threshold
  - And large energy QCD can be used to show $F \sim 1/q^2$
- What we do: explicitly modify the outer function to correct the behaviour in the two limits

What's new?

- We have to reproduce the ρ pole in our parameterisation
- Hard to see how a polynomial expansion can fit this behaviour



Belle 0805.3773

What’s new?

- It can be shown that the pole is on the second Riemann sheet
  - So at a z_r value outside the unit disk
- “As known from general principles of quantum field theory”
  - Caprini, Grinstein, Lebed 2017
  - Grinstein & Lebed 2015

The new look

$$F = \frac{W}{\phi} \frac{1}{z - z_r} \frac{1}{z - z_r^*} \sum_i b_i z^i$$

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- Physical pole at z_r ✅

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- Finite at threshold ✅

The new look

$$F = \frac{W}{\phi} \frac{1}{z - z_r} \frac{1}{z - z_r^*} \sum_i b_i z^i$$

- Physical pole at z_r ✅
- Finite at threshold ✅
- Correct large energy limit ✅

What about the dispersive bound

- Dispersive bound is of the form $\int |\phi F|^2 \leq 1$
- With the standard form ($F = \frac{1}{\phi} \sum_i \alpha_i z^i$), the bound nicely simplifies to $\sum_i |\alpha_i|^2 \leq 1$
- But with our form (with explicit pole factors), doesn't simplify like that
  - We were unable to come up with a form that preserves the simple dispersive bound expression

New parameterisation

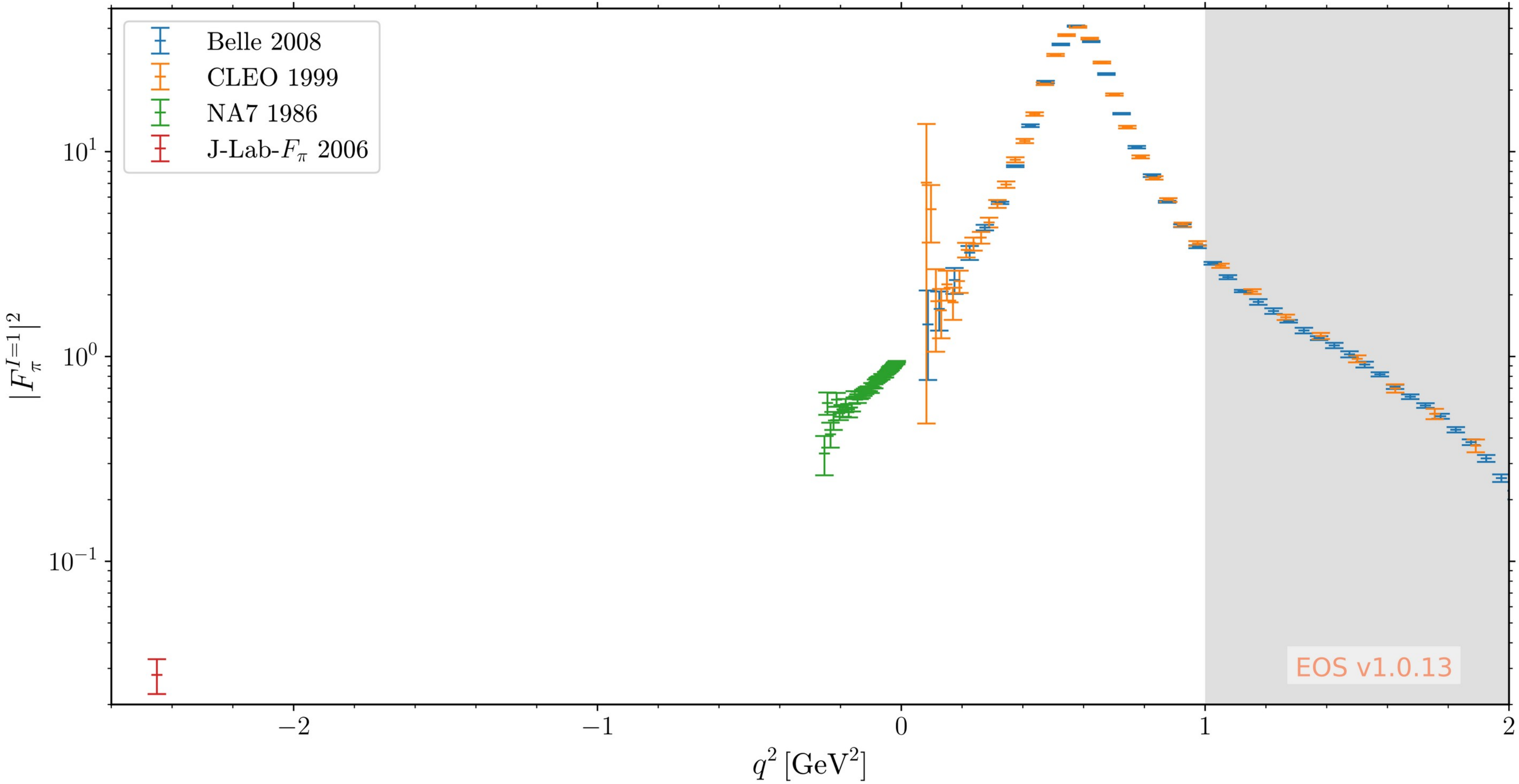
- $F = \frac{W}{\phi} \frac{1}{z - z_r} \frac{1}{z - z_r^*} \sum_i b_i z^i$
  - Physical pole at z_r ✅
  - Finite at threshold ✅
  - Correct large energy limit ✅
  - Dispersive bound on parameters not manifest 😥
- Let’s feed in some data and see what we get

Results

Pion form factor data

- $\tau \to \pi\pi\nu$: depends on $|F_\pi(q^2 > t_+)|^2$, measured by Belle and CLEO
- $\pi - \mathrm{H}$ scattering: depends on $|F_\pi(q^2 < 0)|^2$, measured by NA7
- $ep \to e\pi$: depends on $|F_\pi(q^2 \ll 0)|^2$, measured by JLAB F_π

Pion form factor data



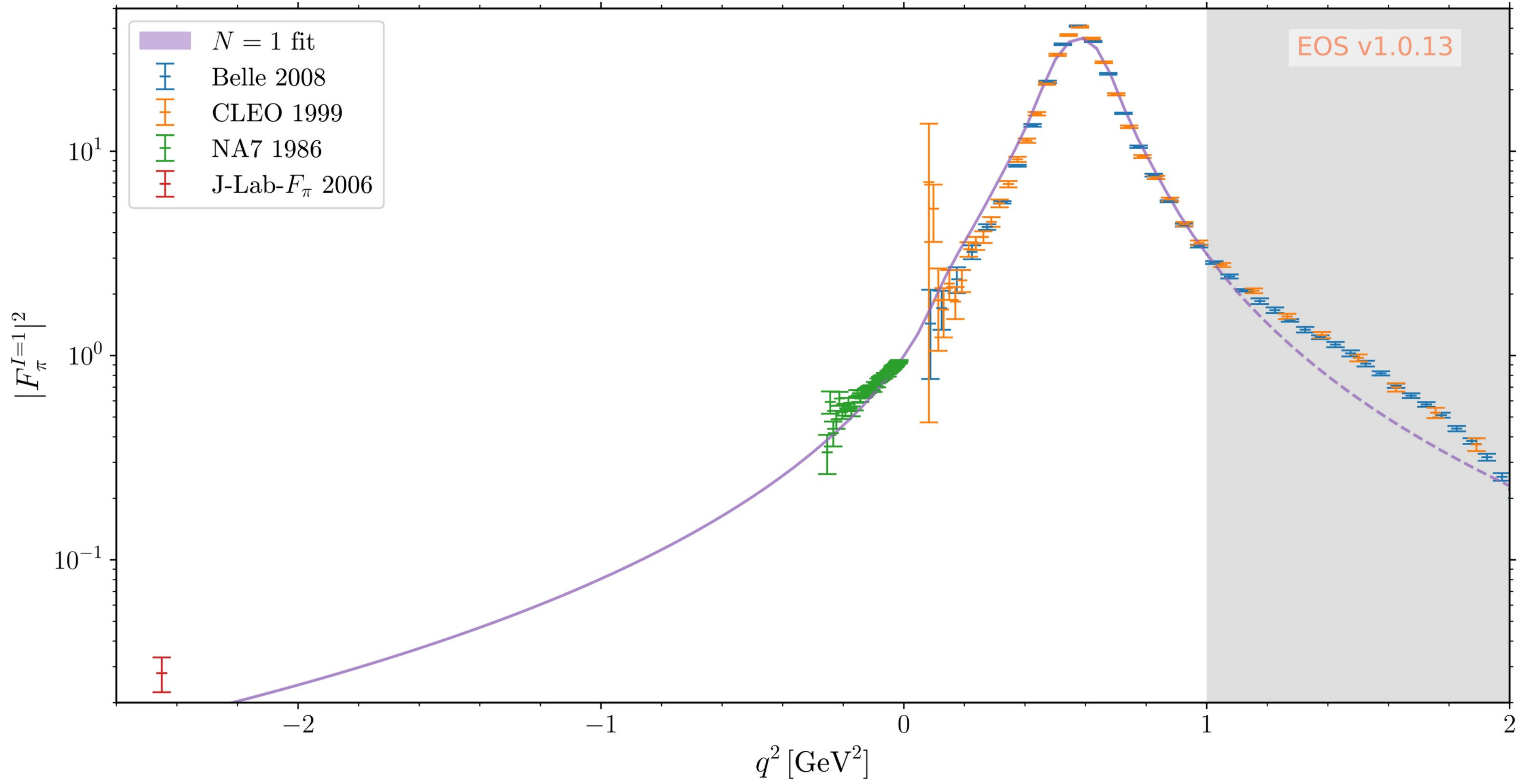
Imposing conditions on our FF

- For equal mass quarks, our current is a conserved current $\Rightarrow F(0) = 1$
- Angular momentum conservation tells us that near threshold $\mathrm{Im}\, F(q^2 \sim t_+) \sim (q^2 - t_+)^{3/2}$
- Impose these by fixing two expansion coefficients

Fits to different order

- With our constraints, if we truncate at order N, we have N–1 free expansion parameters
- Plus two parameters from ρ pole – mass and width
- So for order N truncation, we have a total of N+1 parameters to fit to our 94 data points

Fits to different order



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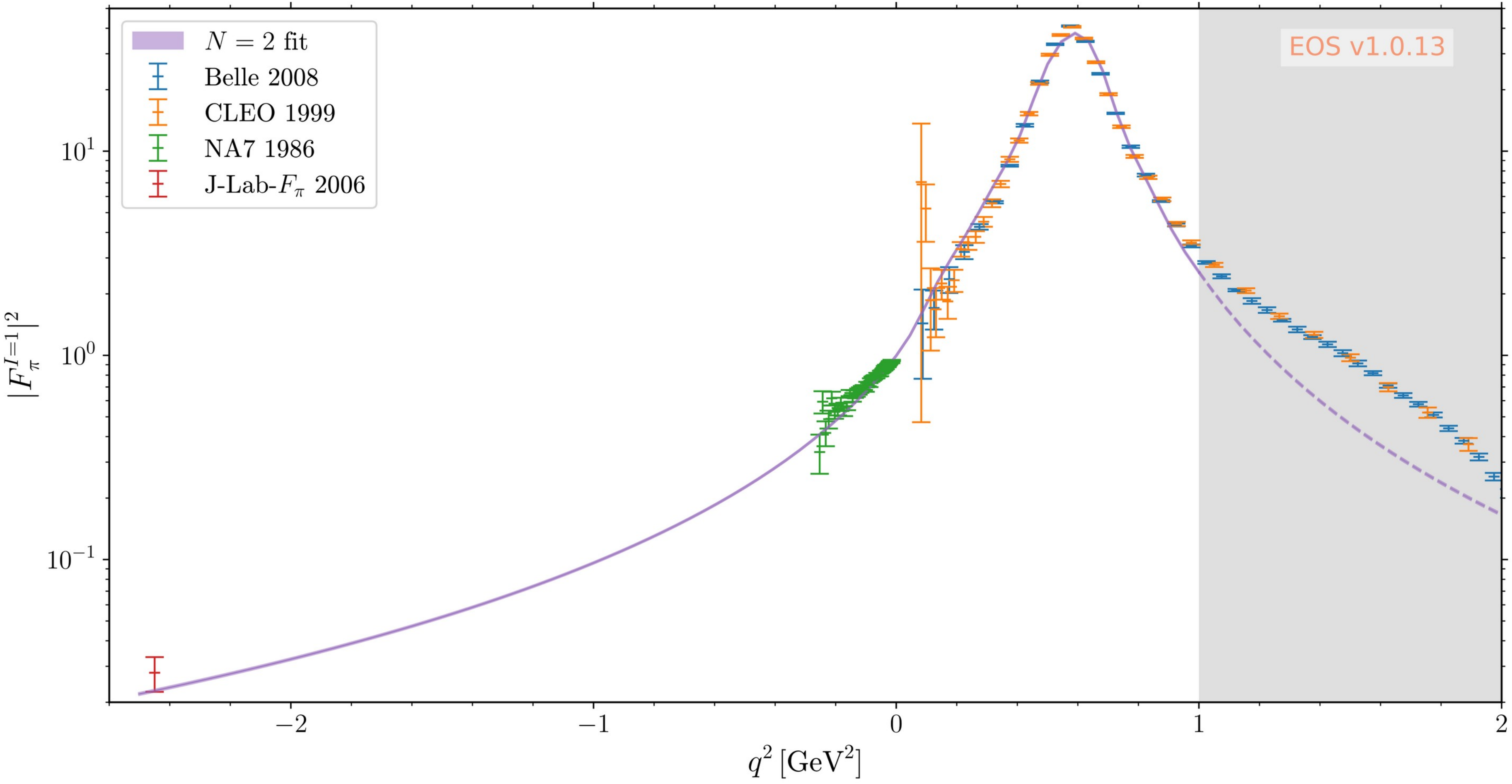

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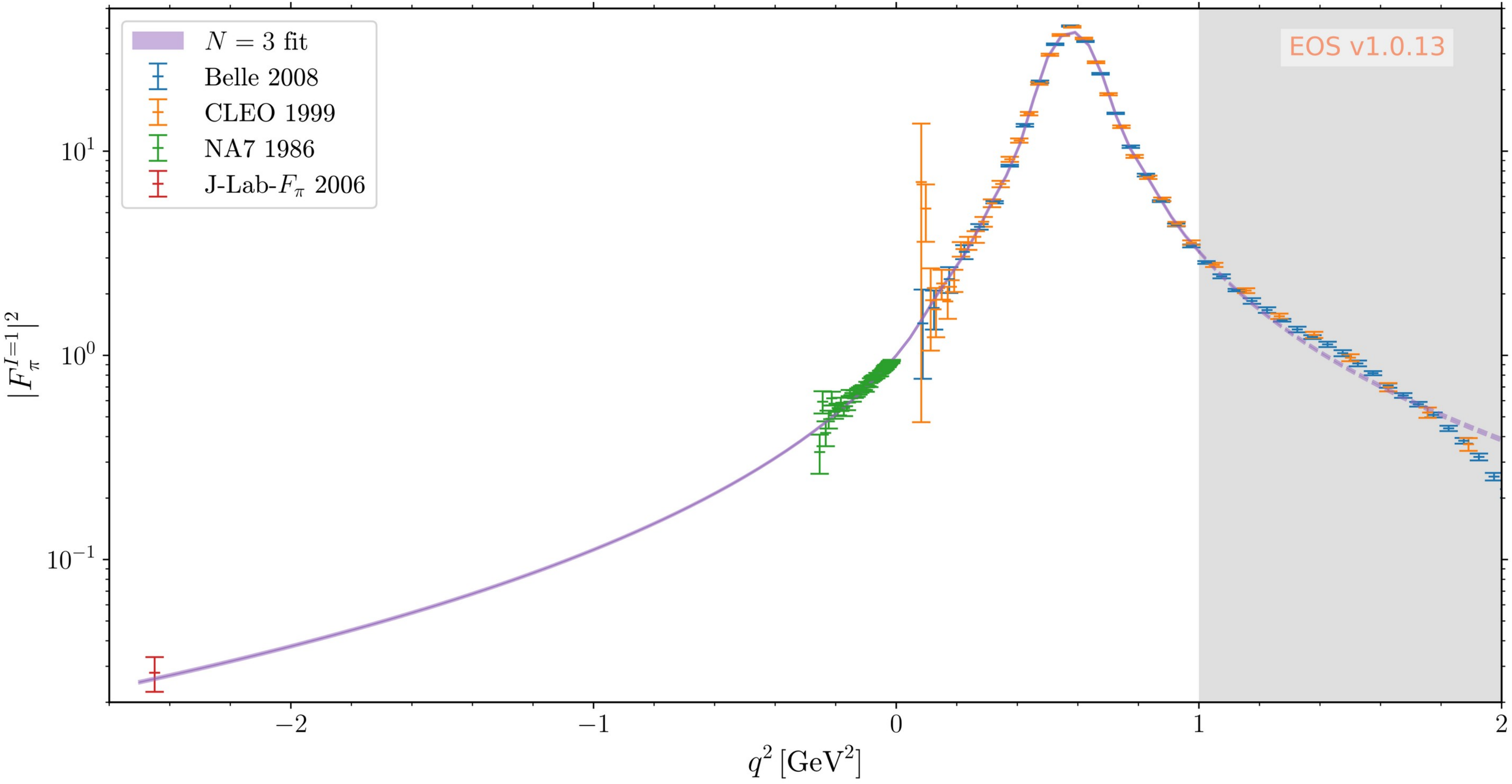

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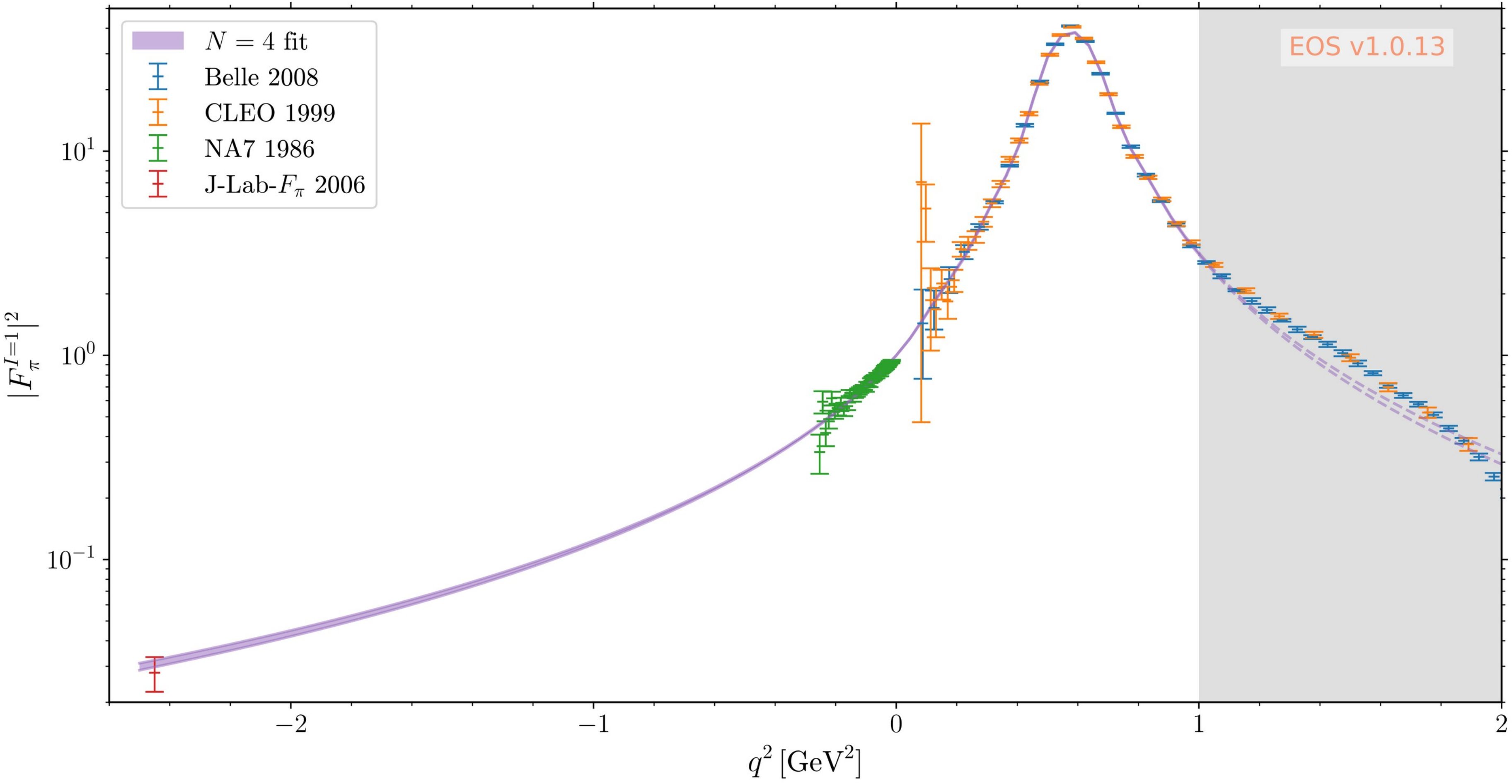

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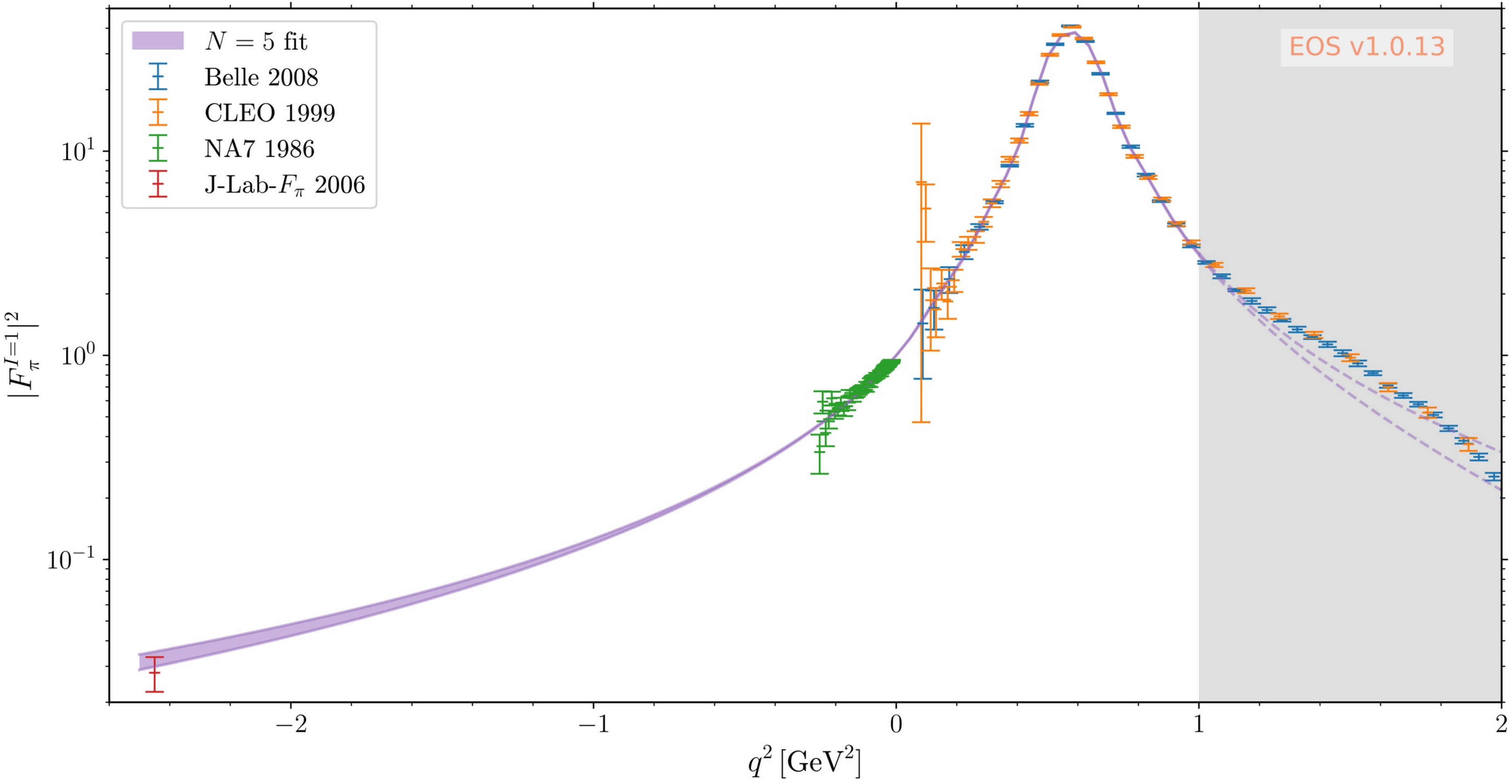
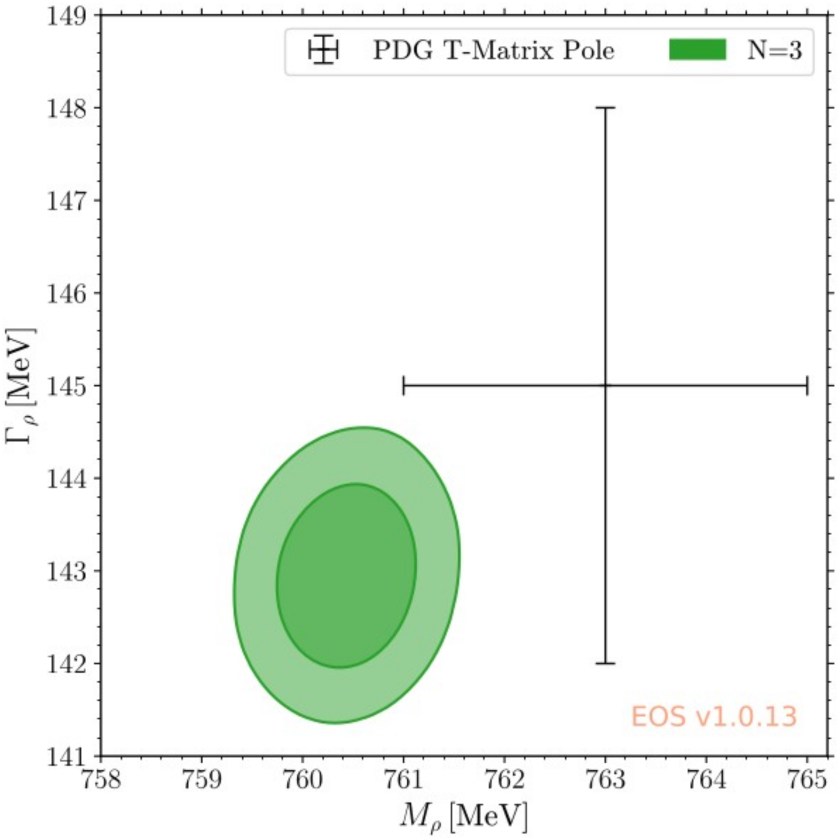


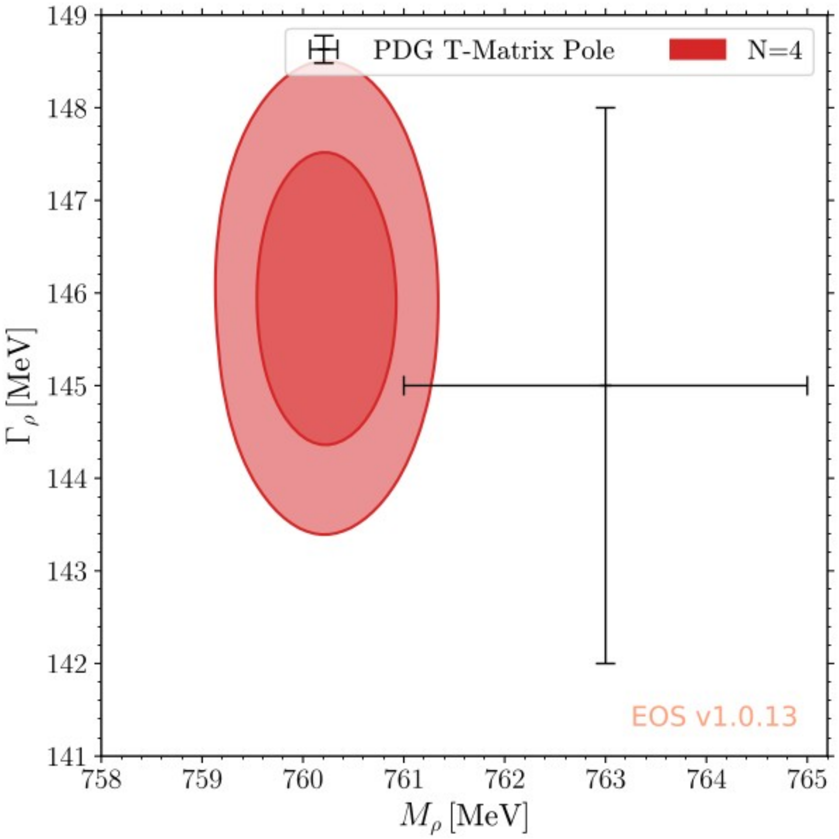
ρ pole parameters

- We extract the ρ mass and width from our fit



ρ pole parameters

- We extract the ρ mass and width from our fit
- Stable under increasing order of the expansion



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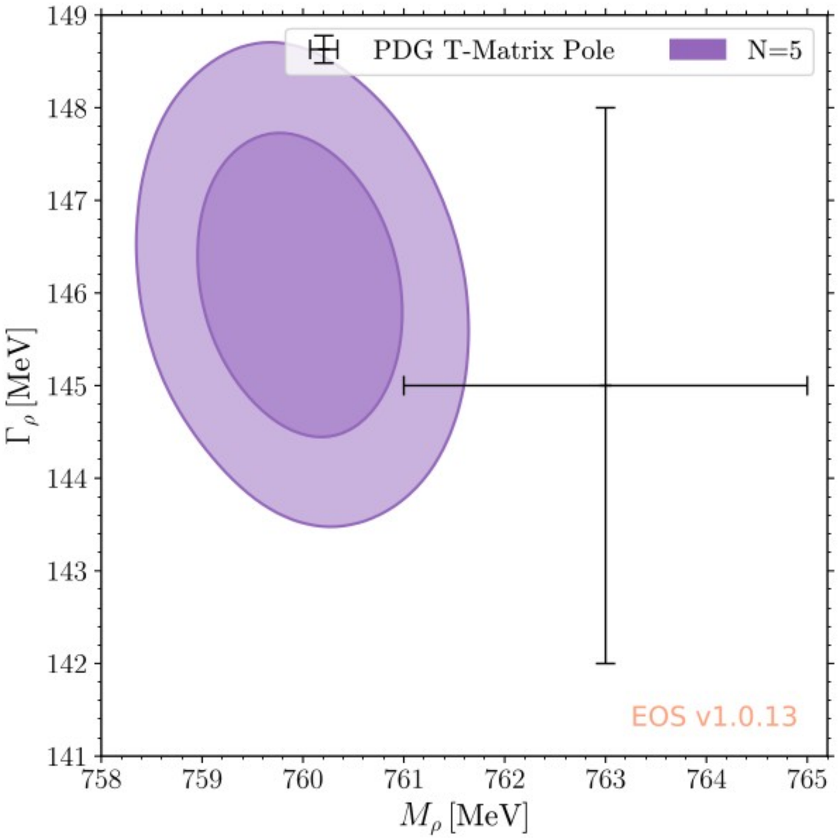


ρ pole parameters

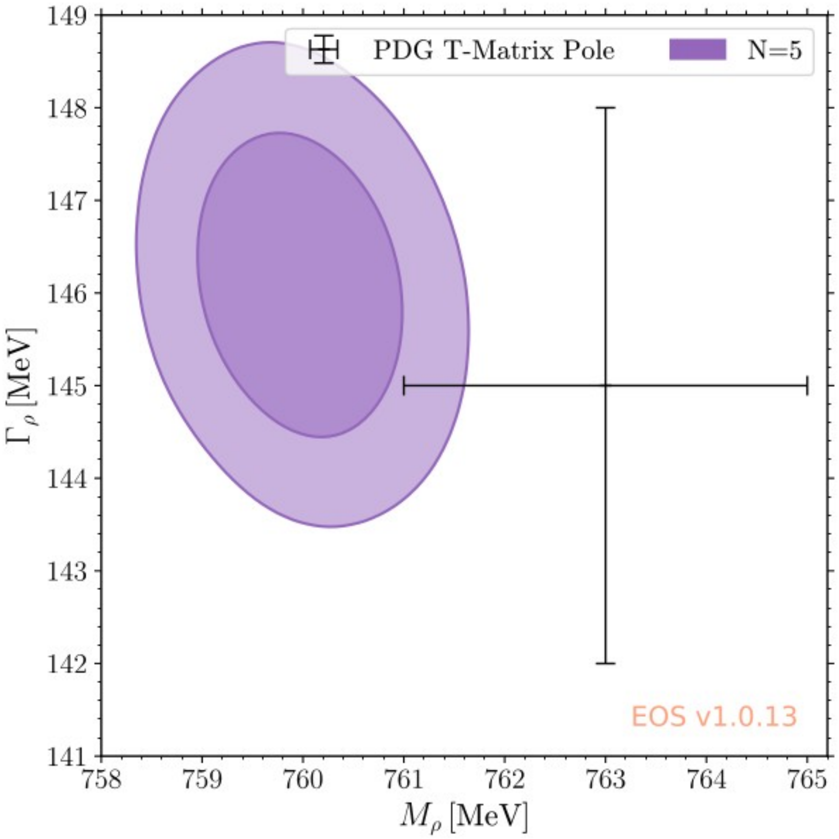
- Our N=5 fit gives

$$M_\rho = (760.0 \pm 0.6)\,\mathrm{MeV}$$

$$\Gamma_\rho = (146.1 \pm 0.9)\,\mathrm{MeV}$$

for ρ pole location

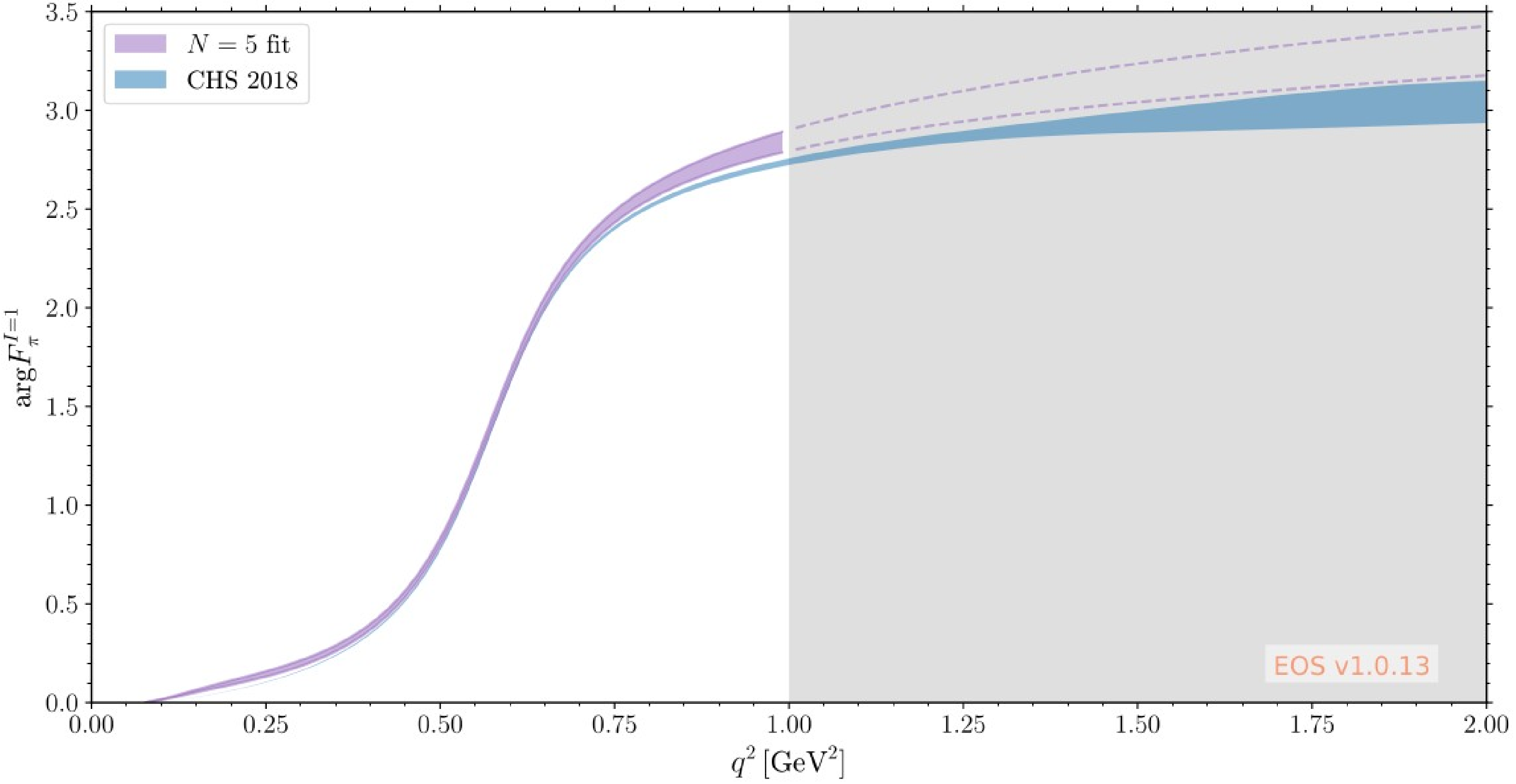
- Reasonable agreement with PDG which comes from other methods



Alternative analyses

- Using analyticity, one can determine the magnitude if you know the phase on the branch cut up to infinity
- Extract the phase up to inelastic threshold, model the phase in the inelastic region

Comparison to other work

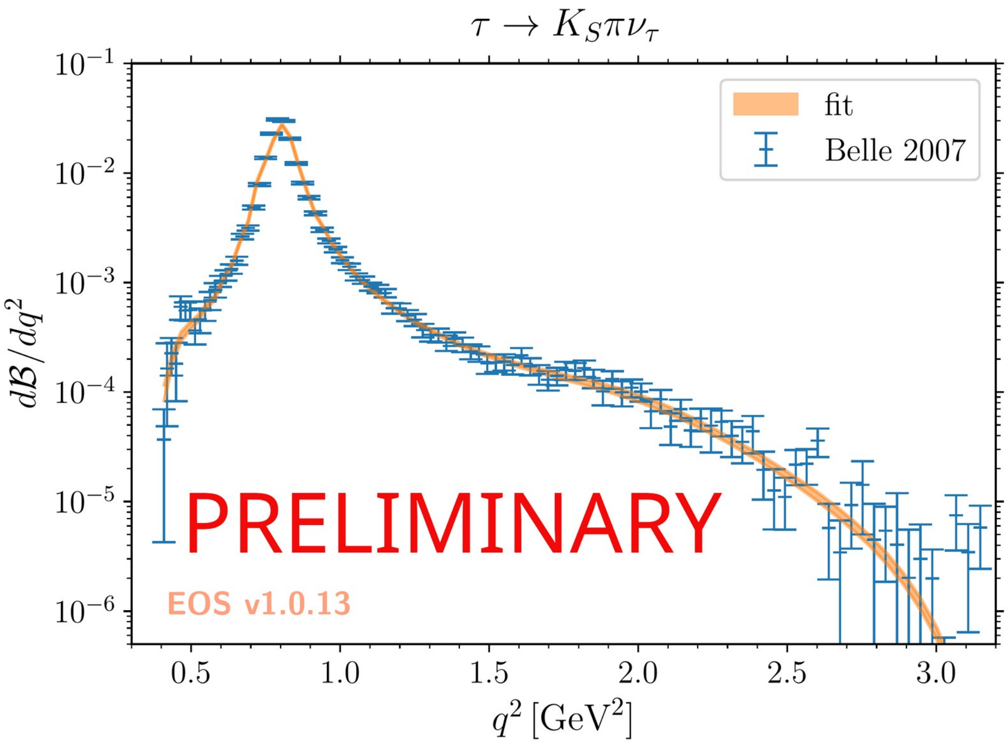


Future outlook and summary

Going forward

- Now we have successful proof of concept, we are working on the $K \to \pi$ case
  - Allows a fit to V_{us}
- Ask me later about Cabibbo angle anomaly

Sneak peak at $\tau \to K\pi\nu$



- Now 2 form factors (scalar and vector), each with two above threshold resonances: $K^*(890), K^*(1410)$, $K_0^*(700), K_0^*(1430)$
- We can fit them!

Summary

- We came up with a new way to parameterise form factors
  - Valid both above and below threshold, explicitly including resonance poles

Summary

- We came up with a new way to parameterise form factors
- Allows to fit to data from all parts of phase space
  - But unlike other parameterisations, don’t need phase data to infinity

Summary

- We came up with a new way to parameterise form factors
- Allows to fit to data from all parts of phase space
- Proof of concept for pion form factor
  - Clear how to extend to e.g. $K \to \pi$, isospin breaking in pions, …

BACKUP

Experts: why not Blaschke factors?

- For subthreshold poles, one can multiply by a Blaschke factor
  - $B(z; z_r) = \frac{z - z_r}{1 - z z_r^*}$
- Which removes a pole at $z = z_r$
- Above threshold, $|B(z; z_r)| = 1$ so dispersive bound simplifies better

Why not Blaschke factors?

- Could we not write our form factor as
  - $F = \frac{W}{\phi} \frac{1}{B(z;z_r)} \frac{1}{B(z;z_r^*)} \sum b_i f_i$?
- Since this still has the pole at the ρ ?

Why not Blaschke factors?

- Could we not write our form factor as
  - $F = \frac{W}{\phi} \frac{1}{B(z;z_r)} \frac{1}{B(z;z_r^*)} \sum b_i f_i$?
- Since this still has the pole at the ρ ?
- No! Now it has two zeros at $z = 1/z_r^{(*)}$, which are inside the unit circle
  - While in general some FFs are known not to have zeros on first Riemann sheet ❌

Constraints on FF

- Want $F(0) = 1$
- Define $z_0 = z(q^2 = 0)$
  - So $1 = F(0) = \frac{W(z_0)}{\phi(z_0)} \sum_i b_i z_0^i$
- One condition on the b_i

Constraints on FF

- We want $\mathrm{Im}\, F(q^2 \sim t_+) \sim (q^2 - t_+)^{3/2}$
- Note $z(t_+) = -1$, $z + 1 \propto (q^2 - t_+)^{1/2}$
- Expand f around -1:
  - $F(z \sim -1) \sim a + b(z-1) + c(z-1)^2 + d(z-1)^3 + \cdots$
  - $F(q^2 \sim t_+) \sim a + B(q^2 - t_+)^{1/2} + C(q^2 - t_+)^1 + D(q^2 - t_+)^{3/2}$
- Impose $0 = df/dz|_{z=-1} = b$ <= another condition

Pion form factor data

- Data exists on the pion FF in several different kinematic regions
- From NA7 paper:

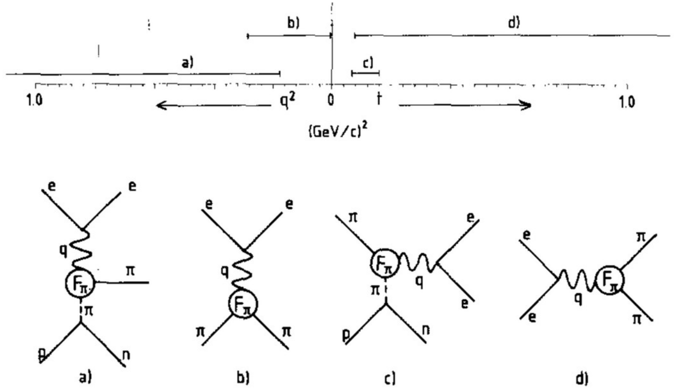
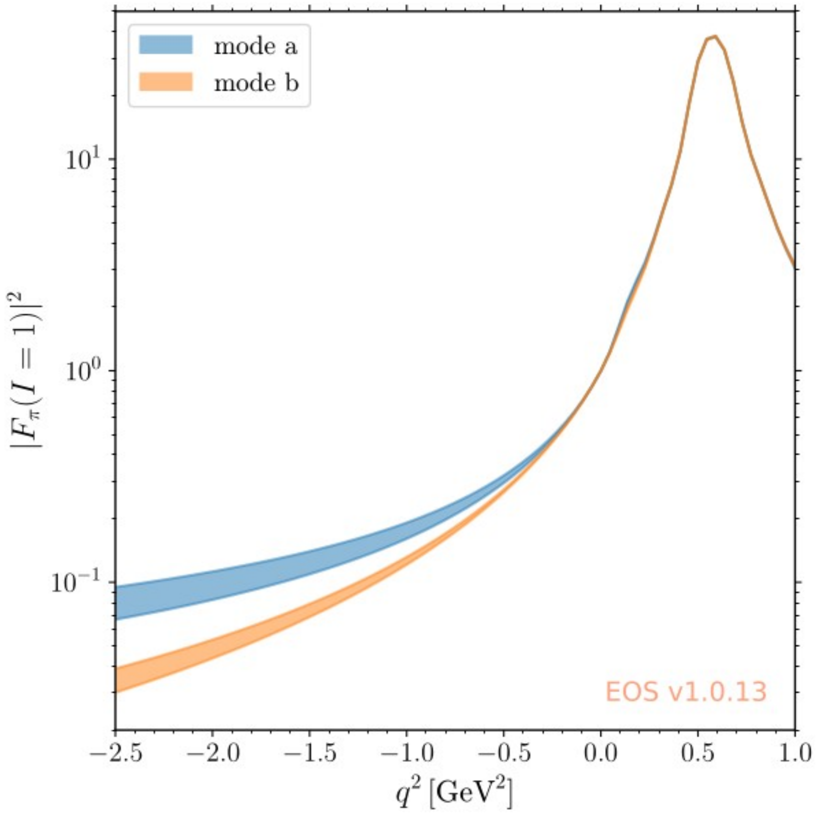
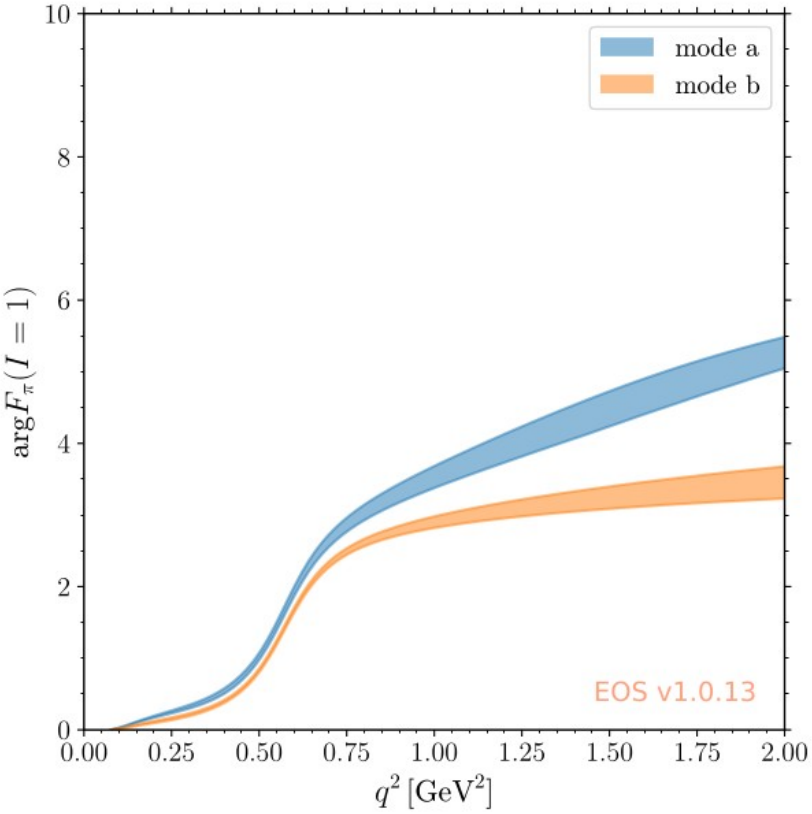


Fig. 1. Data on the squared modulus of F_π for $|t| < 1(\mathrm{GeV}/c)^2$ from the reactions: (a) electroproduction [1]; (b) direct πe scattering [2–4]; (c) inverse electroproduction [5]; and (d) e^+e^- annihilation [6–9]. The horizontal bar (b) indicates the range of our experiment.

Zeros on real axis





Fitting semi-leptonic data

- $F = \frac{1}{\phi} \sum_i \alpha_i z^i$
- For semi-leptonic region, $z(q^2)$ is real and $|z| < 1$
  - E.g. for $B \to D$, can choose t_0 such that $|z| < 0.04$, for $B \to K$, $|z| < 0.3$
- The sum converges, and quickly