# A new look for the pion form factor

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(based on 2410.13764 with Kubis, Reboud, van Dyk)

- Why are we interested in form factors?
- Overview of dispersive bounds
- What about above threshold data?
- Results
- Future outlook and summary

# Why are we interested in form factors?

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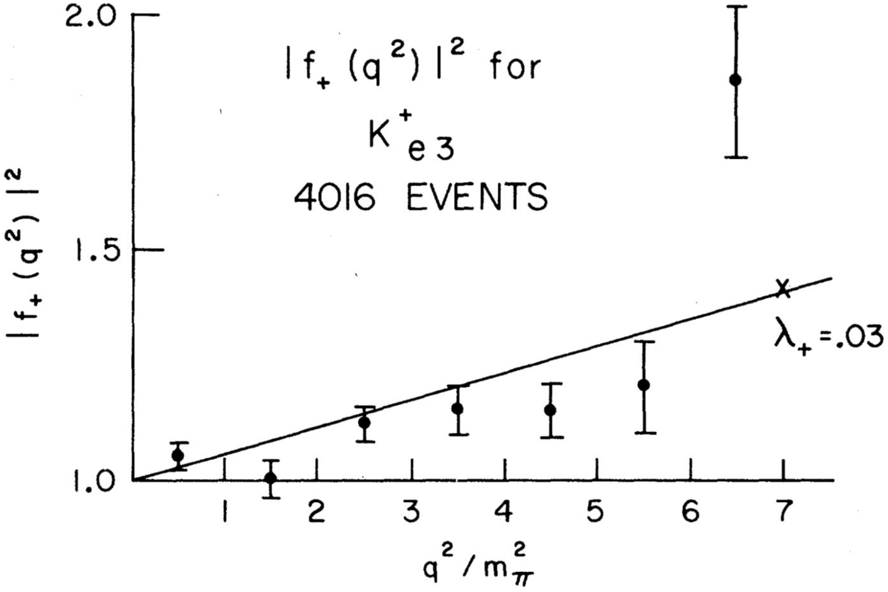
- Semi-leptonic decays are very interesting
  - E.g. for determining CKM elements, but also potential BSM
- Consider $K \to \pi$ which is used to extract $V_{us}$
- But $\tau \to K\pi\nu$ should also give access to $V_{us}$

# Why are we interested in form factors?

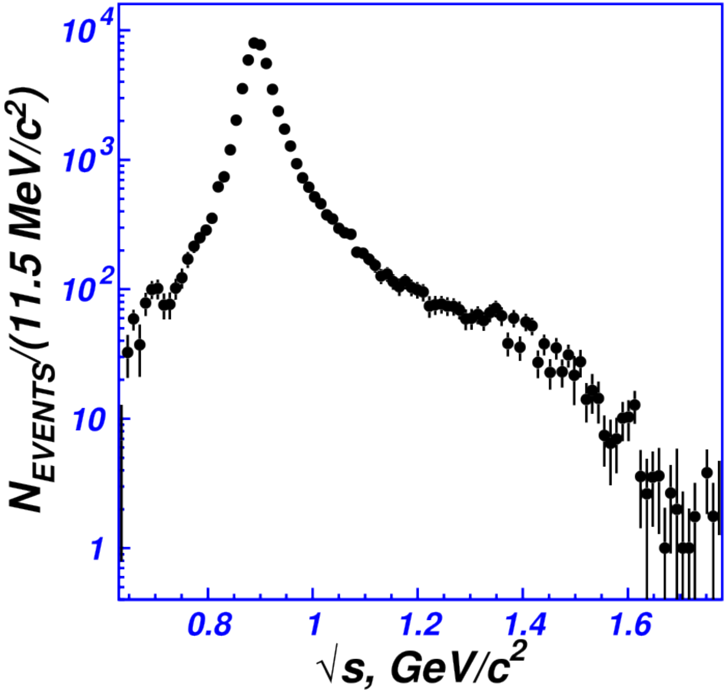
$$K \to \pi e \nu$$




Chiang, Rosen, Shapiro, Handler, Olsen, Pondrom 1972

$$\tau \to K \pi \nu$$

Belle 0706.2231



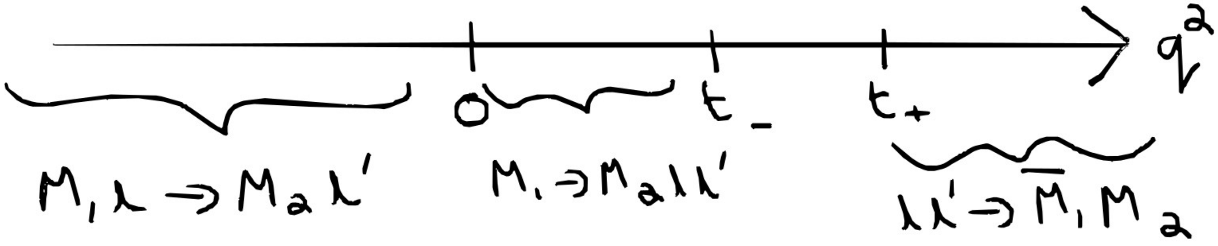
# What is a form factor?

- Hadronic quantities
- $\langle M_2(p_2)|j|M_1(p_1)\rangle \sim F(q^2 = (p_1 - p_2)^2)$
  - $q^2 < 0 : M_1 \ell \to M_2 \ell'$
  - $0 < q^2 < t_- : M_1 \to M_2 \ell \ell'$
  - $q^2 > t_+ : \ell \ell' \to \bar{M}_1 M_2$

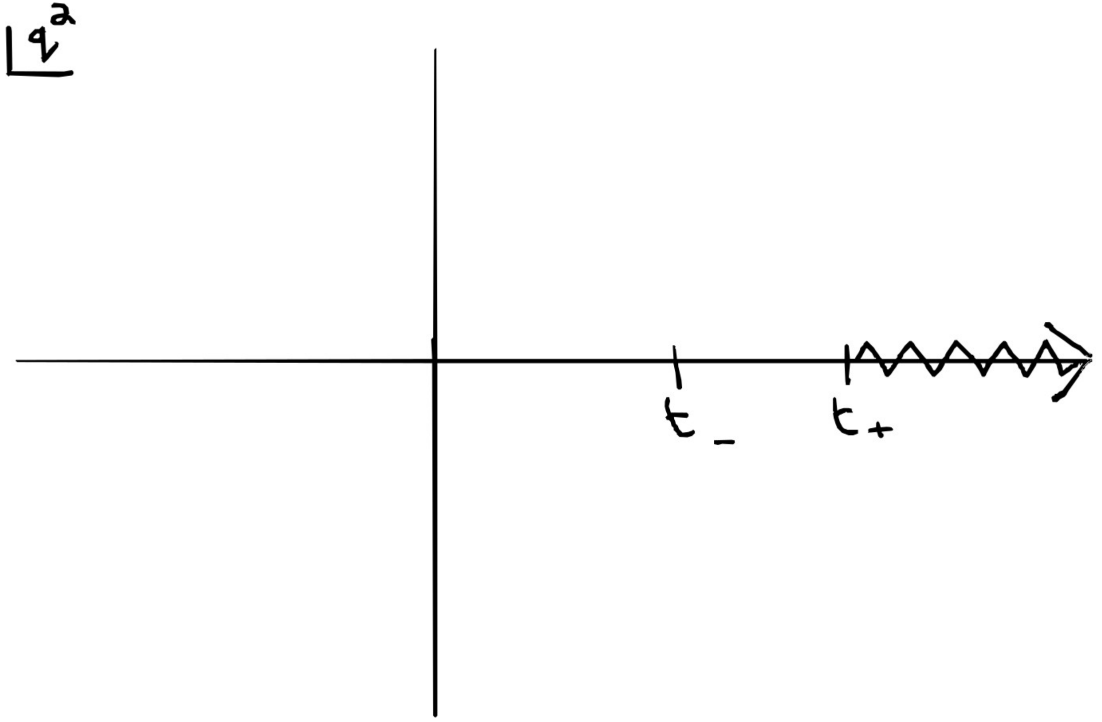
$$t_\pm = (m_1 \pm m_2)^2$$

$$\langle \overline{M_1} M_2 | j | 0 \rangle$$

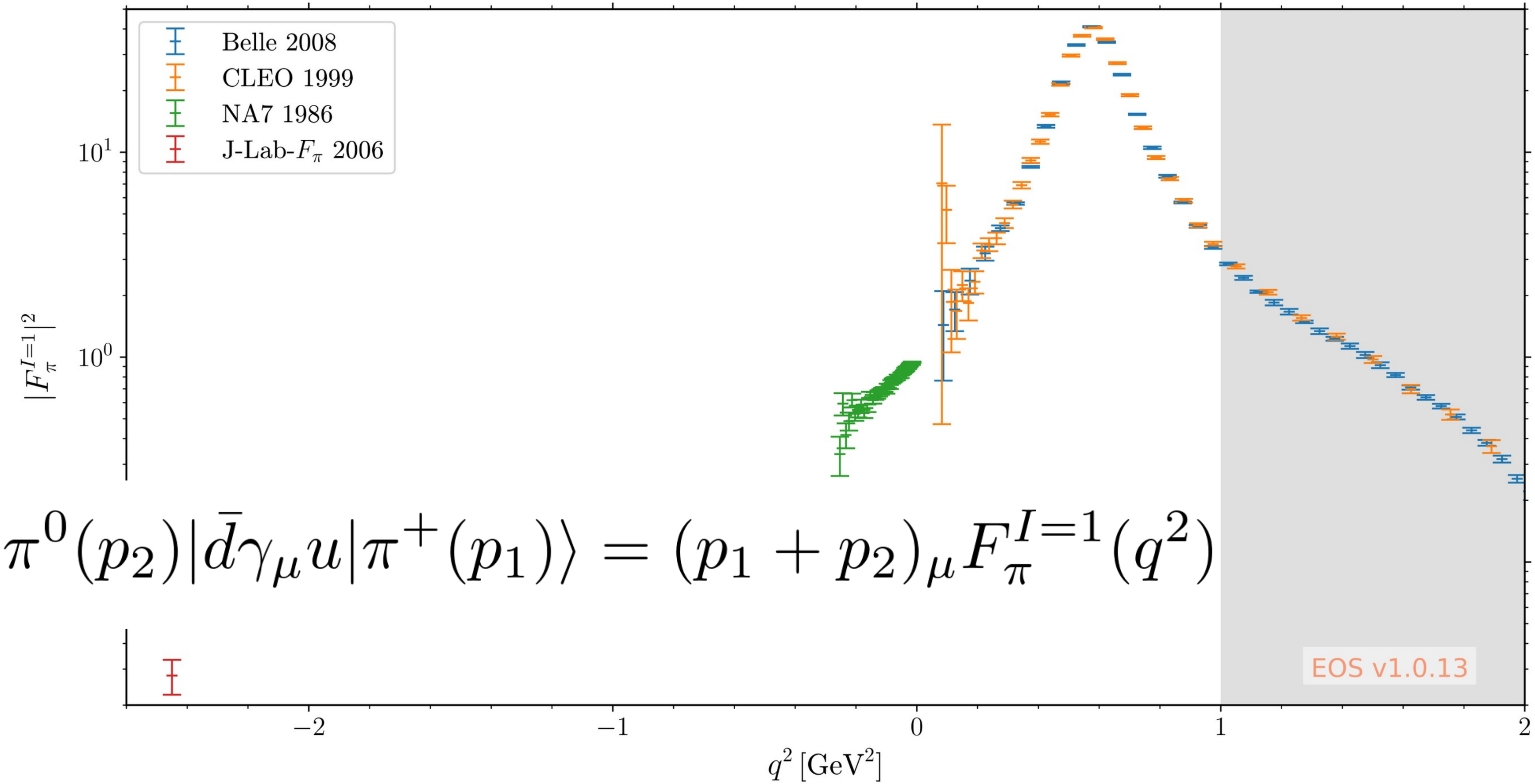
# What is a form factor?



# What is a form factor?



# Simple case: pion form factor

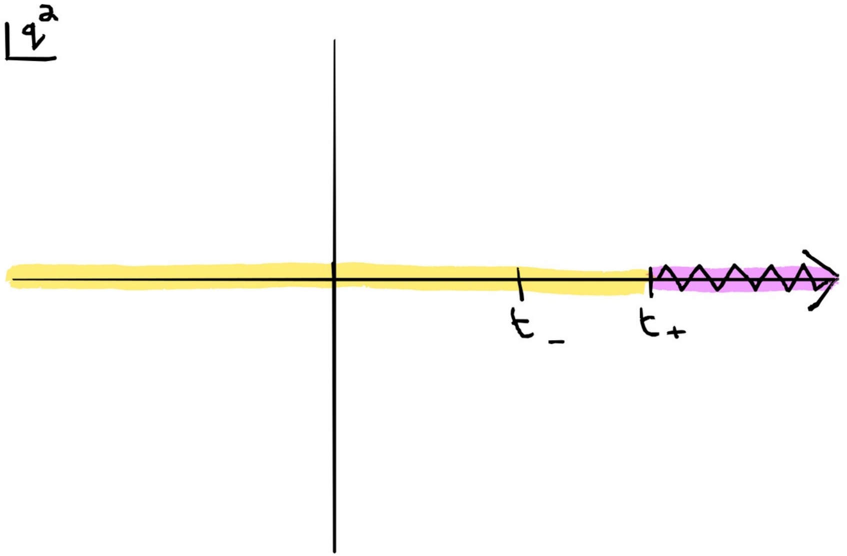


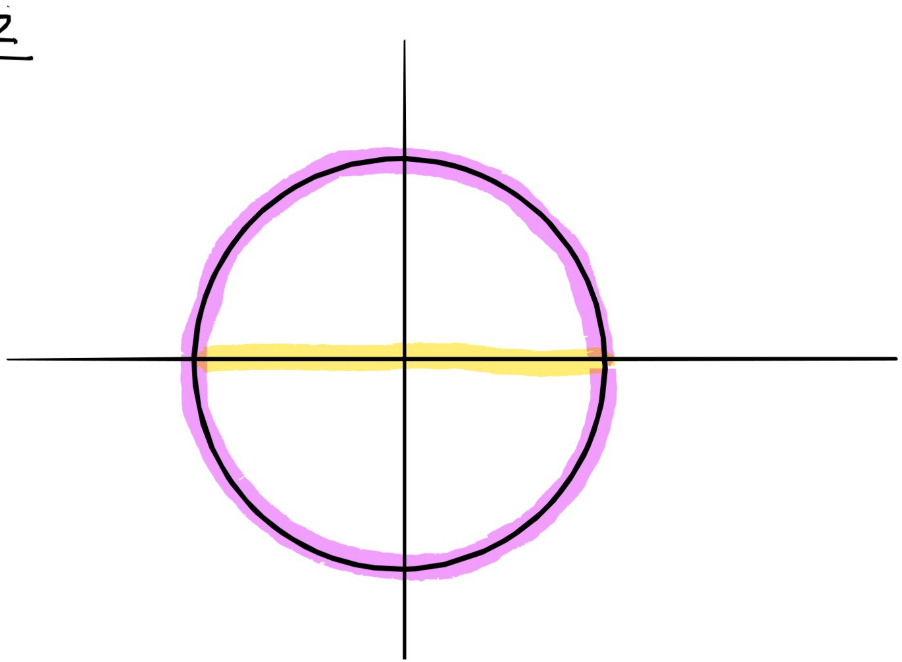

$$\frac{1}{\sqrt{2}}\langle\pi^0(p_2)|\bar{d}\gamma_\mu u|\pi^+(p_1)\rangle = (p_1+p_2)_\mu F_\pi^{I=1}(q^2)$$

# How do we describe form factors?

- Common parameterisation uses a conformal mapping from $q^2$ plane to $z$
  - First used for form factors in Meiman (1963, JETP), Okubo (1971, PRD)
  - Made famous by BGL parameterisation

# Conformal mapping



# How do we describe form factors?

- Common parameterisation uses a conformal mapping from $q^2$ plane to $z$
- Why is this useful?
- Need to understand dispersive bounds…

# Overview of dispersive bounds

# Dispersive bounds in 1 slide

- Write $e^- \bar{\nu} \to \bar{u}d$ in both inclusive (i.e. perturbative quark level) and exclusive (sum over meson states) way
- Inclusive ≥ Exclusive
  - Inclusive we calculate in QCD using OPE
  - Exclusive depends on form factor

# Dispersive bounds in 3 slides

- Consider $\Pi^{\mu\nu}(q^2) =$ ~~~O~~~
- $\Pi$ is analytic, except on the positive real axis, where there are poles from resonances
- Use Cauchy to write $\Pi(q^2) = \oint dt \frac{\Pi(t)}{t-q^2}$
- Analytic structure means we can write this as $\Pi(q^2) = \int_{t_+}^{\infty} \frac{\mathrm{Im}\,\Pi(t)}{t-q^2}$

# Dispersive bounds in 3 slides

- $\Pi(q^2) = \int_{t_+}^{\infty} \frac{\operatorname{Im}\Pi(t)}{t-q^2}$
- For $q^2$ very large and negative, LHS is calculable using an OPE
- While imaginary part related to on-shell intermediate states

# Dispersive bounds in 3 slides

- $\Pi = \int dt \frac{\mathrm{Im}\,\Pi}{t-q^2} \sim \int dt \frac{1}{t-q^2} \int_{\mathrm{P.S.}} \sum_X \langle 0|j|X\rangle\langle X|j^\dagger|0\rangle$
- Look just at two particle terms: $X = P_1 \bar{P}_2$
- By crossing symmetry, this is just our form factor!
- RHS is a sum of positive terms, so we can drop terms and just replace the equality with an inequality. This is the basic dispersive bound!

# Simplifying the dispersive bound

- Often we write $F \sim \sum_i \alpha_i f_i$, for some functions $f_i$
- Ideally chosen such that, given our Cauchy integral, the RHS reduces to $\sum_i |\alpha_i|^2$
- So the dispersive bound becomes a simple bound on parameters $\alpha_i$

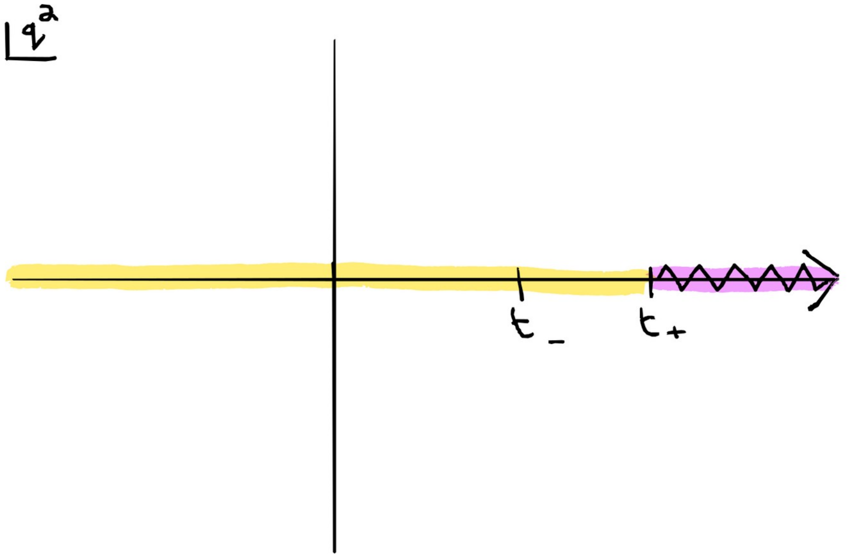
# Simplifying the dispersive bound

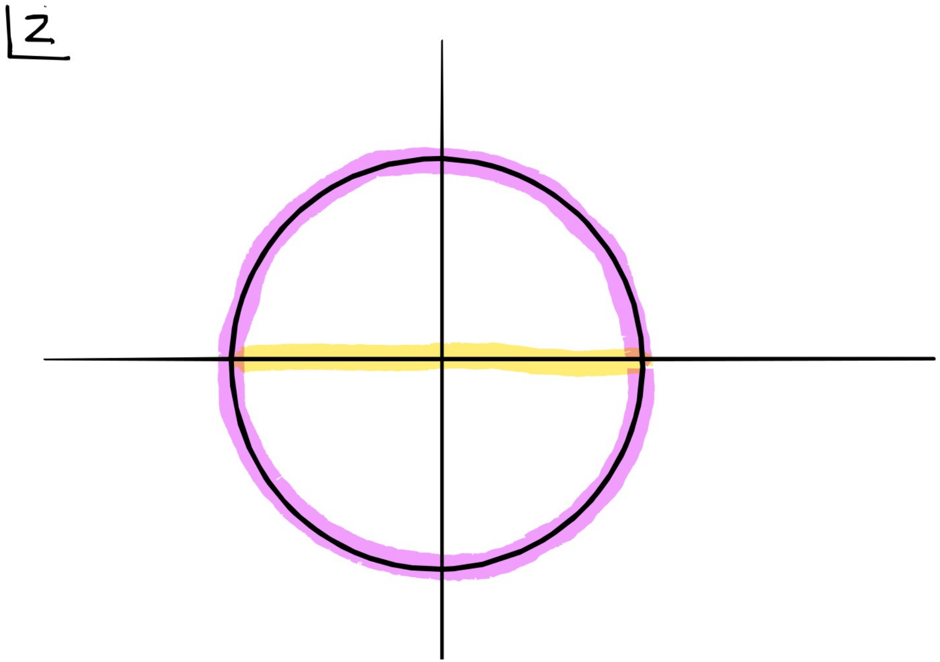
- Can be tricky to choose $f_i$ properly
  - See Danny's work with Méril, Nico, Javier on sub threshold poles 2305.06301 (Gubernari, Reboud, van Dyk, Virto)
- For our analysis of the pion form factor, we did not find a nice choice
  - See later in this talk for what the issues are

# Apply the conformal mapping

- Found $\int_{t_+}^{\infty} dt(\ldots)|F|^2 \leq 1$
- (...) comes from inclusive calculation, Cauchy denominator, plus phase space factors – usually written as $|\phi|^2$ and called the outer function
  - Note the outer function is fixed for any transition
- Now change from $q^2$ to $z$

# Apply the conformal mapping



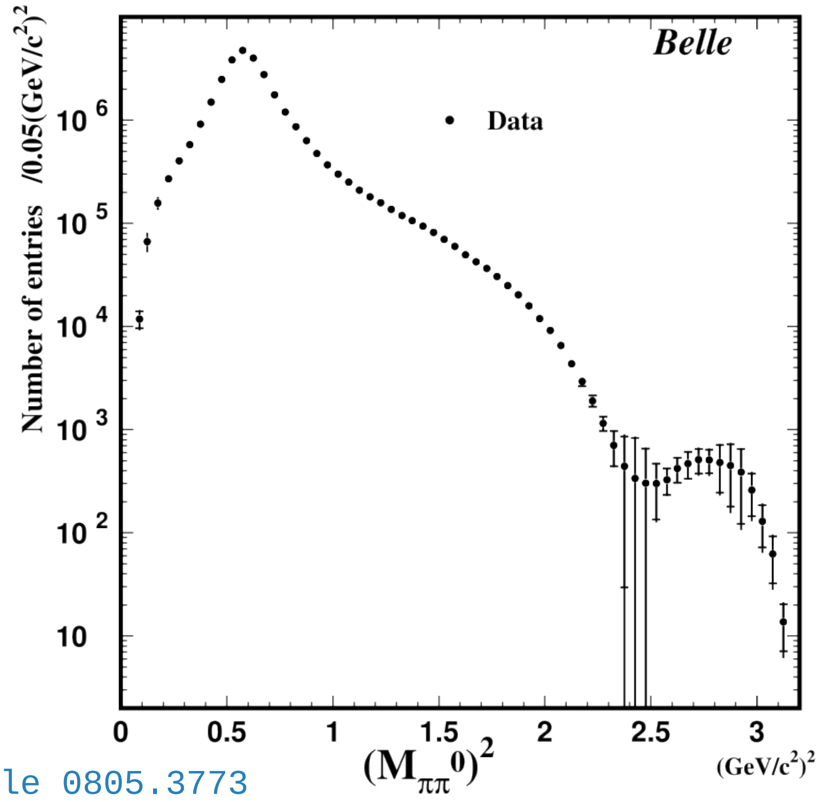



# Apply the conformal mapping

- $\int_{t_+}^{\infty} \to \oint_{|z|=1}$
- Write $F = \frac{1}{\phi} \sum_i \alpha_i z^i$
- Polynomials $z^i$ useful since $\oint_{|z=1|} z^i \bar{z}^j dz = \delta_{ij}$
- Dispersive bound become extremely simple!
- Just $\sum_i |\alpha_i|^2 \leq 1$

# What about above threshold data?

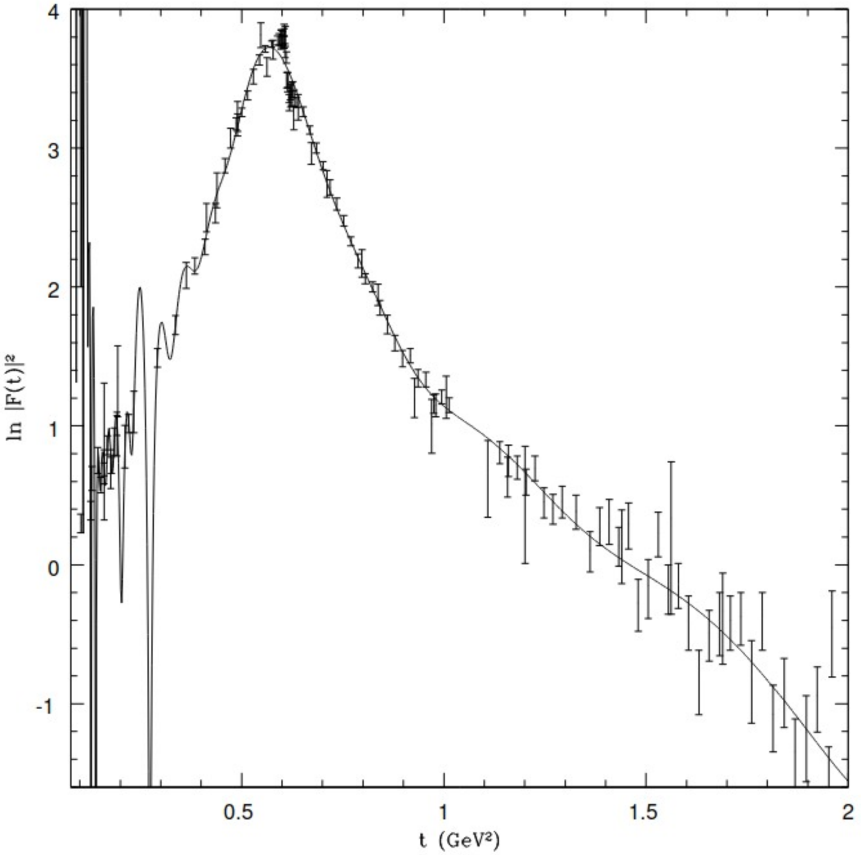
# Data in the above threshold region




Belle 0805.3773

- Can we also use data here as part of the fit?
- In 1998, Buck & Lebed studied this problem

# Data in the above threshold region




Buck, Lebed (hep-ph/9802369)

- Can we also use data here as part of the fit?
- In 1998, Buck & Lebed studied this problem
- They found no, get spurious oscillations near threshold

# What went wrong?

- For $q^2 > t_+$, our expansion parameter has $|z| = 1$!
- Does the sum even converge?
- Yes – see section IV of Buck Lebed 1998
  - Roughly speaking: the form factor has a physically well defined quantity along the cut in $q^2$, Abel's theorem guarantees the series converges to that limit

# Buck Lebed 1998

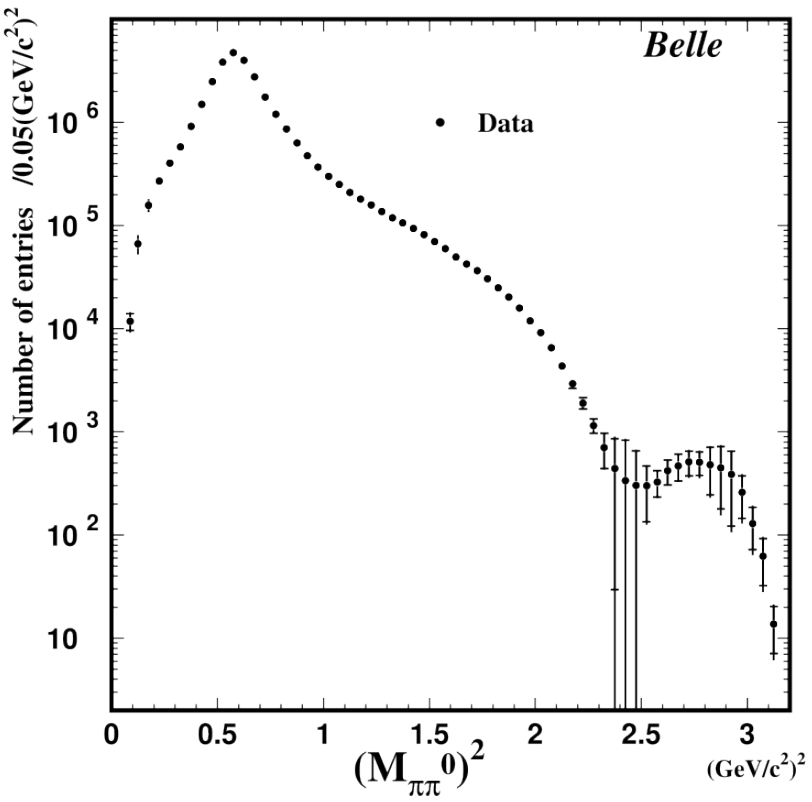
- An issue they discuss is that with $F = \frac{1}{\phi} \sum_i \alpha_i z_i^i$, $F$ picks up two incorrect behaviours from $\phi$
- $\phi$ has a zero at $q^2 = t_+$ => $F$ blows up
- Asymptotic behaviour of $\phi$ as $q^2 \to \infty$ leads to $F(q^2 \to \infty) \sim (q^2)^{1/4}$

# What's wrong? And how do we fix it?

- Neither is physical
  - Experiment tells us $F$ is finite near threshold
  - And large energy QCD can be used to show $F \sim 1/q^2$
- What we do: explicitly modify the outer function to correct the behaviour in the two limits

# What's new?

- We have to reproduce the $\rho$ pole in our parameterisation
- Hard to see how a polynomial expansion can fit this behaviour

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# What's new?

- It can be shown that the pole is on the second Riemann sheet
  - So at a $z_r$ value outside the unit disk
- "As known from general principles of quantum field theory"
  - Caprini, Grinstein, Lebed 2017
  - Grinstein & Lebed 2015

# The new look

$$F = \frac{W}{\phi} \frac{1}{z - z_r} \frac{1}{z - z_r^*} \sum_i b_i z^i$$

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- Physical pole at $z_r$ ✅

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- Finite at threshold ✅

# The new look

$$F = \frac{W}{\phi} \frac{1}{z - z_r} \frac{1}{z - z_r^*} \sum_i b_i z^i$$

- Physical pole at $z_r$ ✅
- Finite at threshold ✅
- Correct large energy limit ✅

# What about the dispersive bound

- Dispersive bound is of the form $\int |\phi F|^2 \leq 1$
- With the standard form ($F = \frac{1}{\phi} \sum_i \alpha_i z^i$), the bound nicely simplifies to $\sum_i |\alpha_i|^2 \leq 1$
- But with our form (with explicit pole factors), doesn't simplify like that
  - We were unable to come up with a form that preserves the simple dispersive bound expression

# New parameterisation

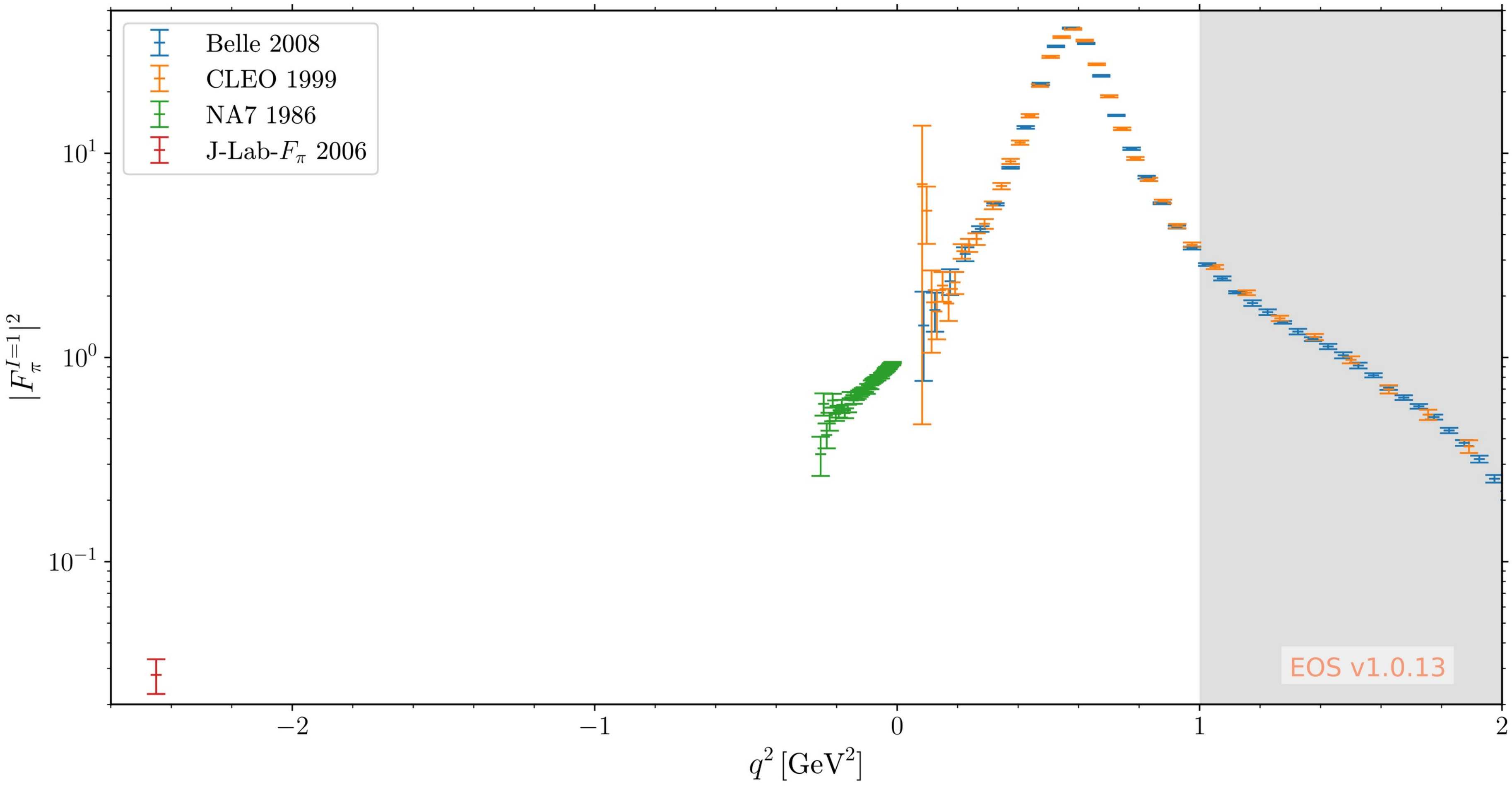
- $F = \frac{W}{\phi} \frac{1}{z - z_r} \frac{1}{z - z_r^*} \sum_i b_i z^i$
  - Physical pole at $z_r$ ✅
  - Finite at threshold ✅
  - Correct large energy limit ✅
  - Dispersive bound on parameters not manifest 😥
- Let’s feed in some data and see what we get

# Results

# Pion form factor data

- $\tau \to \pi\pi\nu$ : depends on $|F_\pi(q^2 > t_+)|^2$, measured by Belle and CLEO
- $\pi - \mathrm{H}$ scattering: depends on $|F_\pi(q^2 < 0)|^2$, measured by NA7
- $ep \to e\pi$ : depends on $|F_\pi(q^2 \ll 0)|^2$, measured by JLAB $F_\pi$

# Pion form factor data



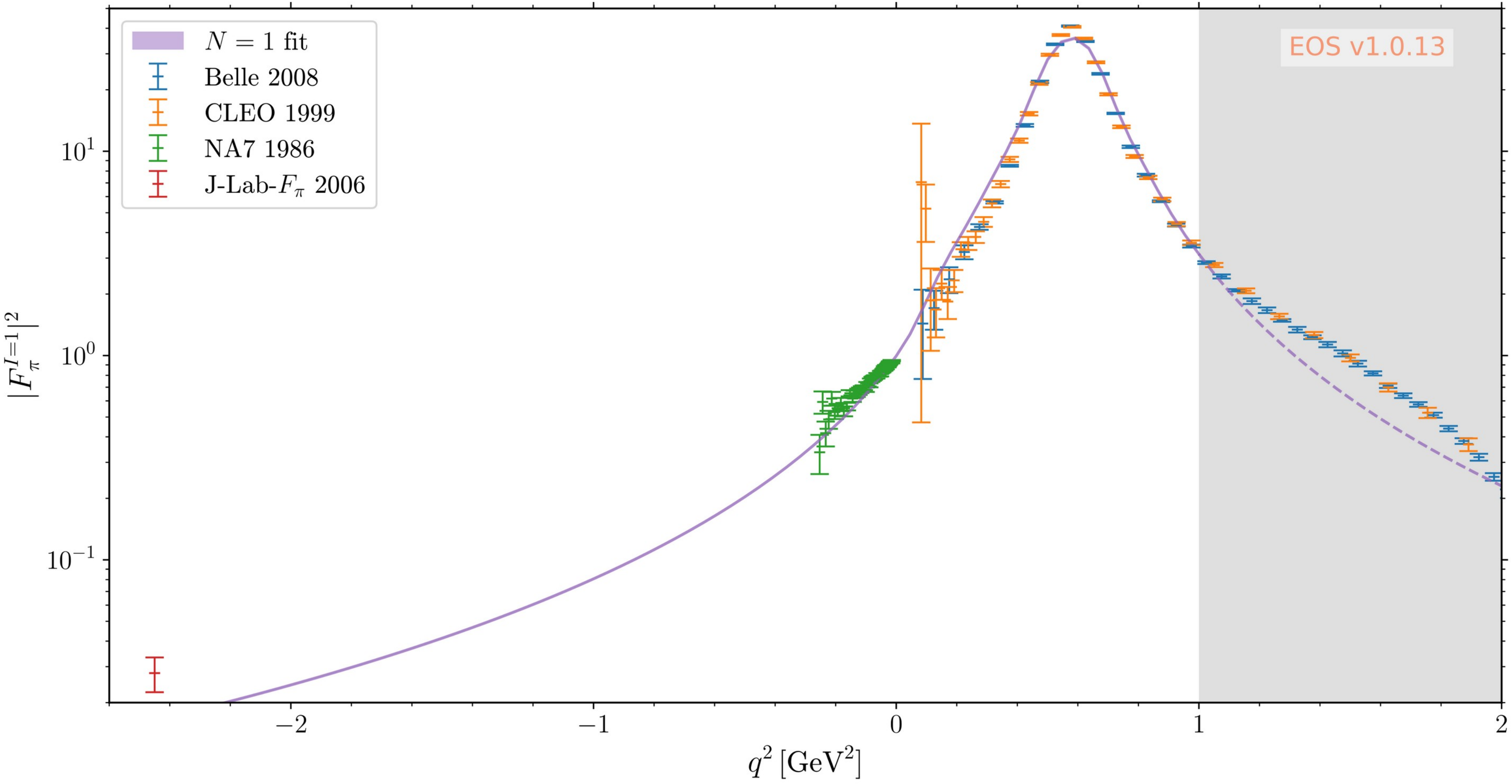
# Imposing conditions on our FF

- For equal mass quarks, our current is a conserved current $\Rightarrow F(0) = 1$
- Angular momentum conservation tells us that near threshold $\operatorname{Im} F(q^2 \sim t_+) \sim (q^2 - t_+)^{3/2}$
- Impose these by fixing two expansion coefficients

# Fits to different order

- With our constraints, if we truncate at order N, we have N-1 free expansion parameters
- Plus two parameters from $\rho$ pole – mass and width
- So for order N truncation, we have a total of N+1 parameters to fit to our 94 data points

# Fits to different order



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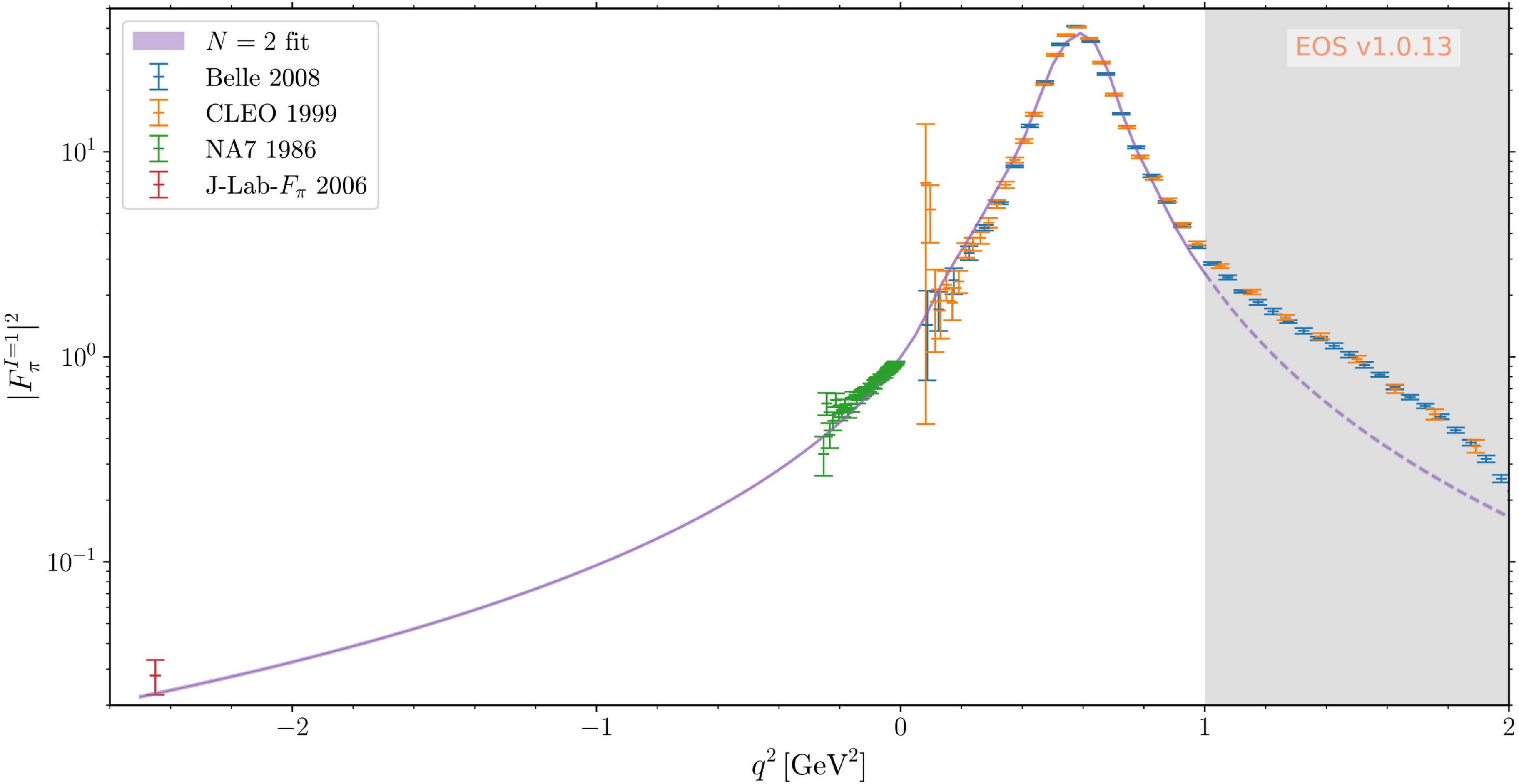

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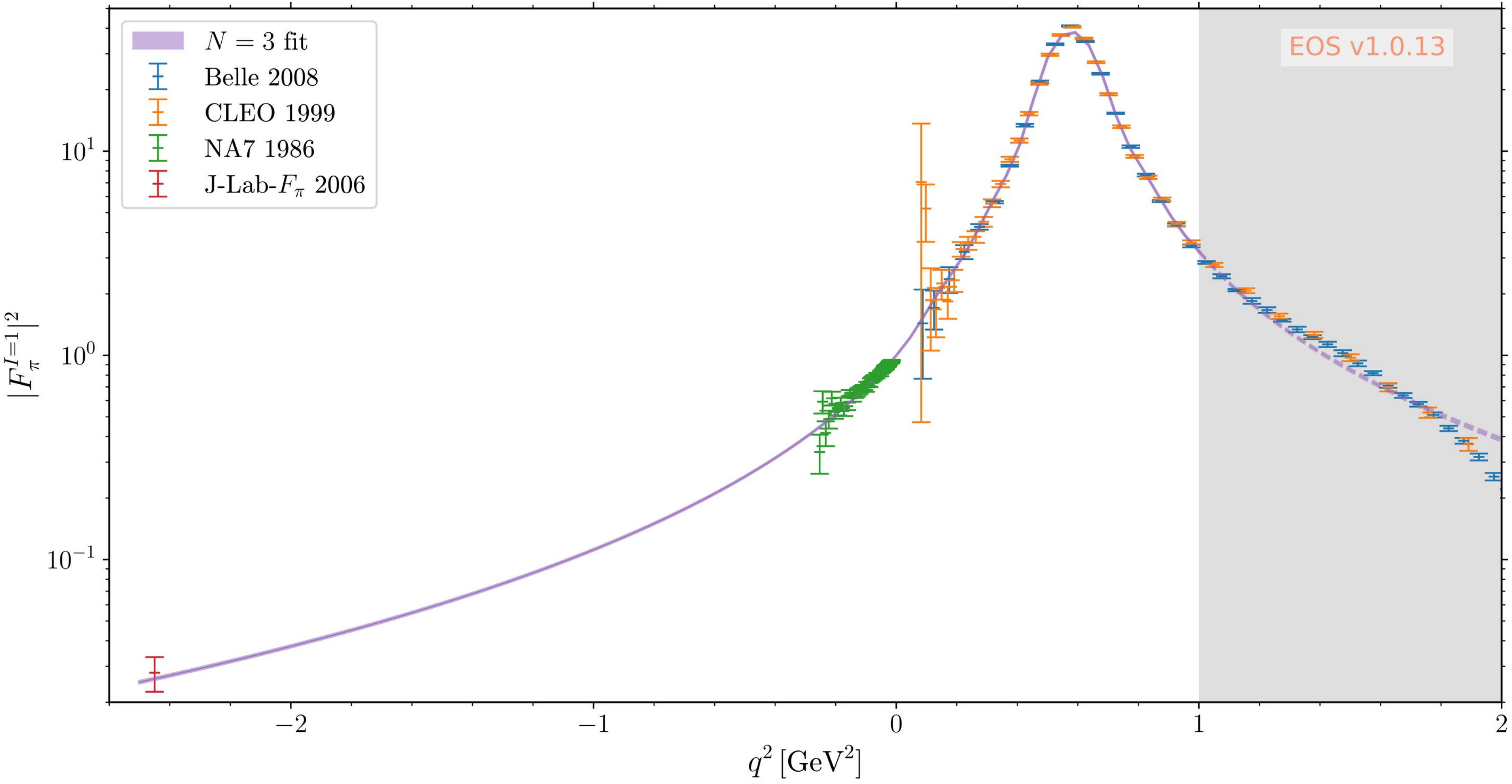

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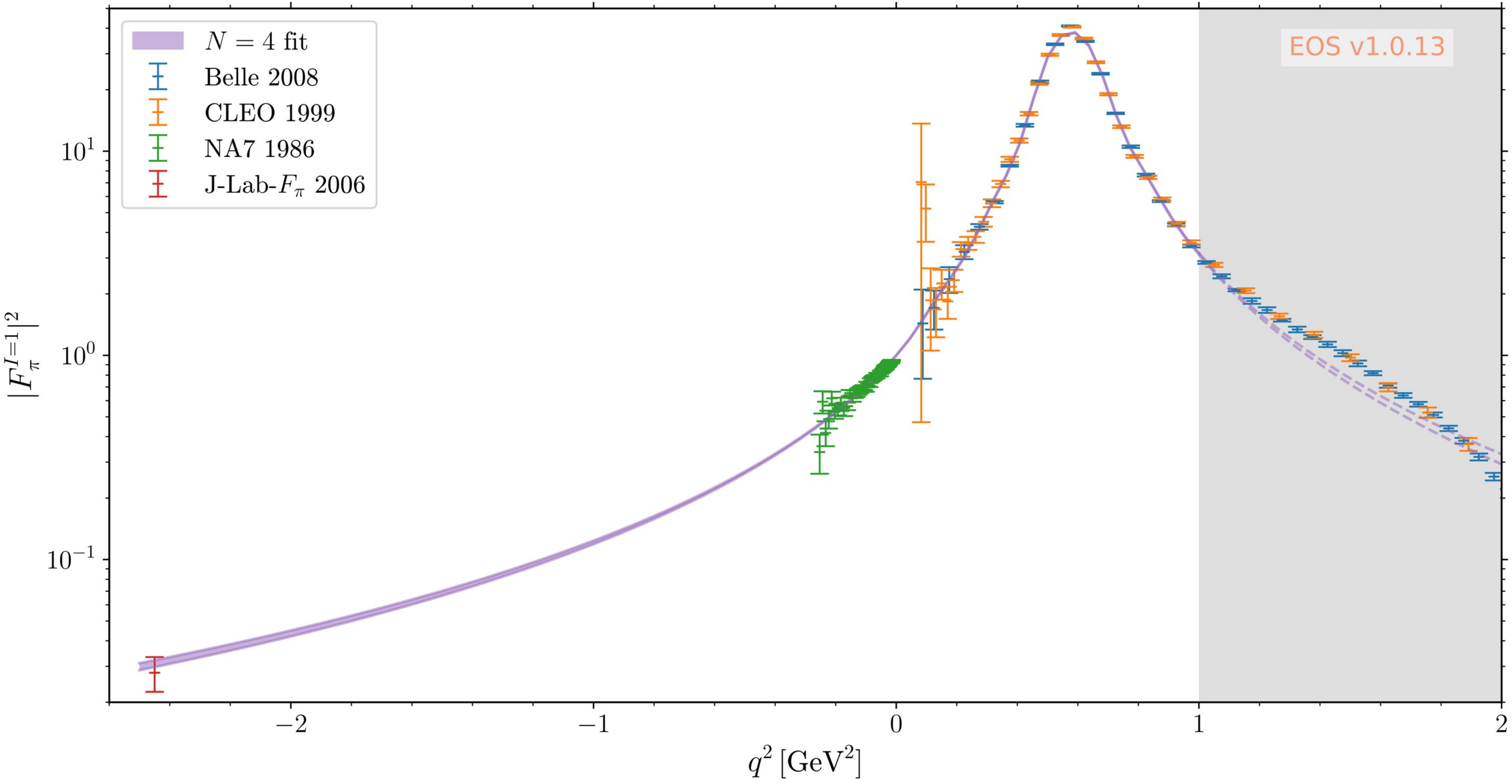

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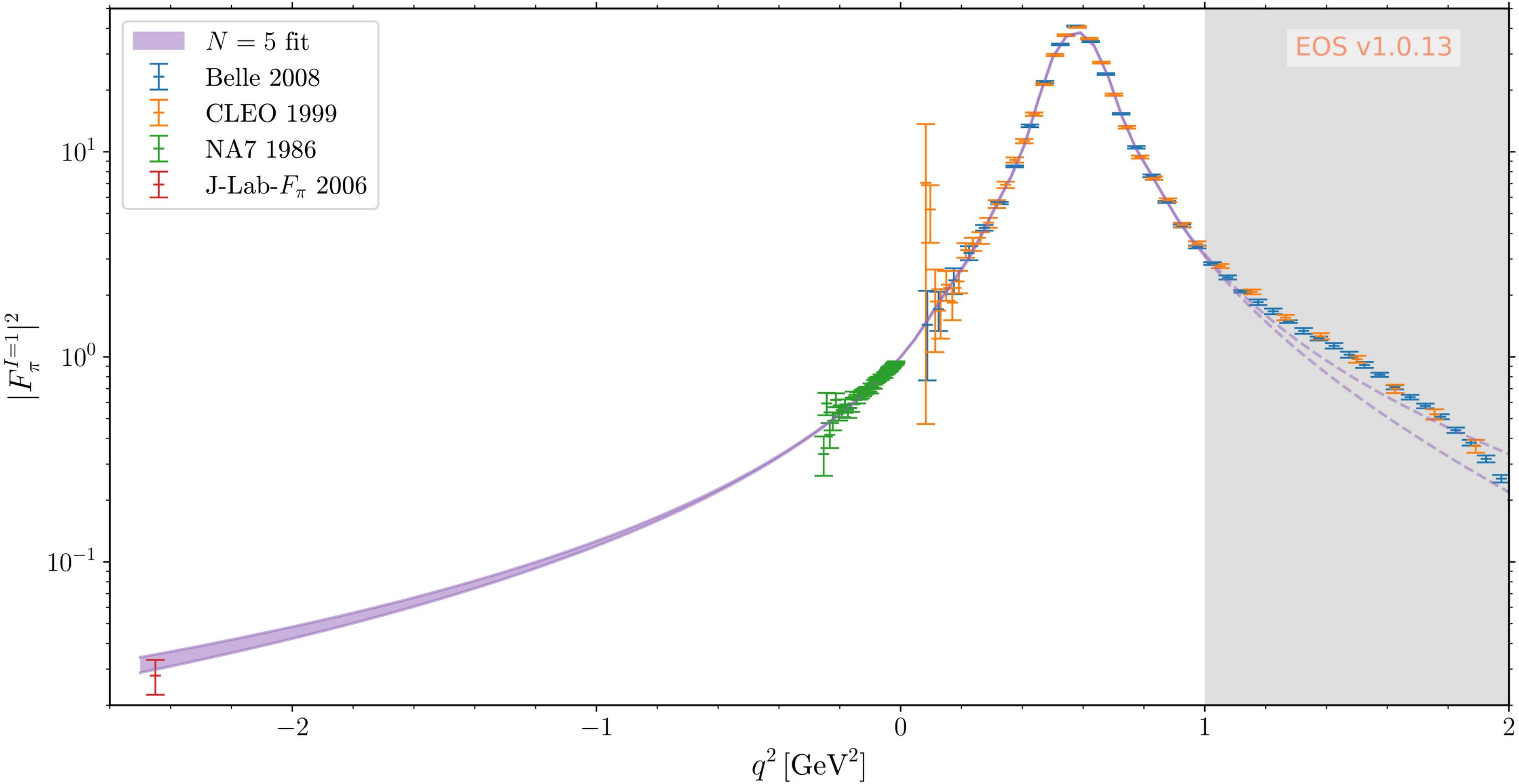
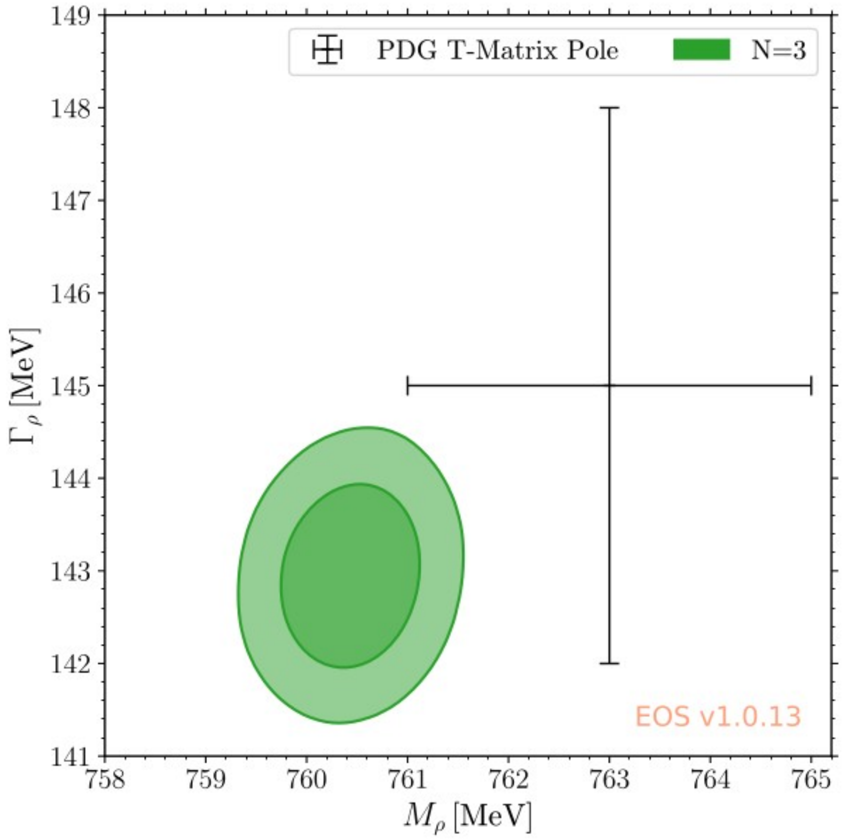


# $\rho$ pole parameters

- We extract the $\rho$ mass and width from our fit



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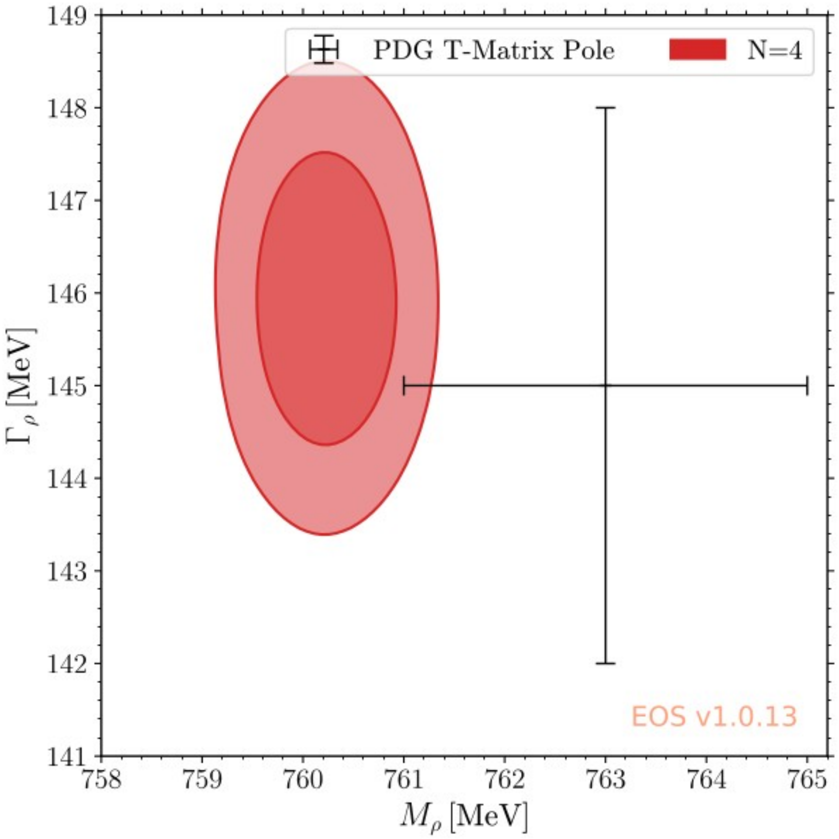

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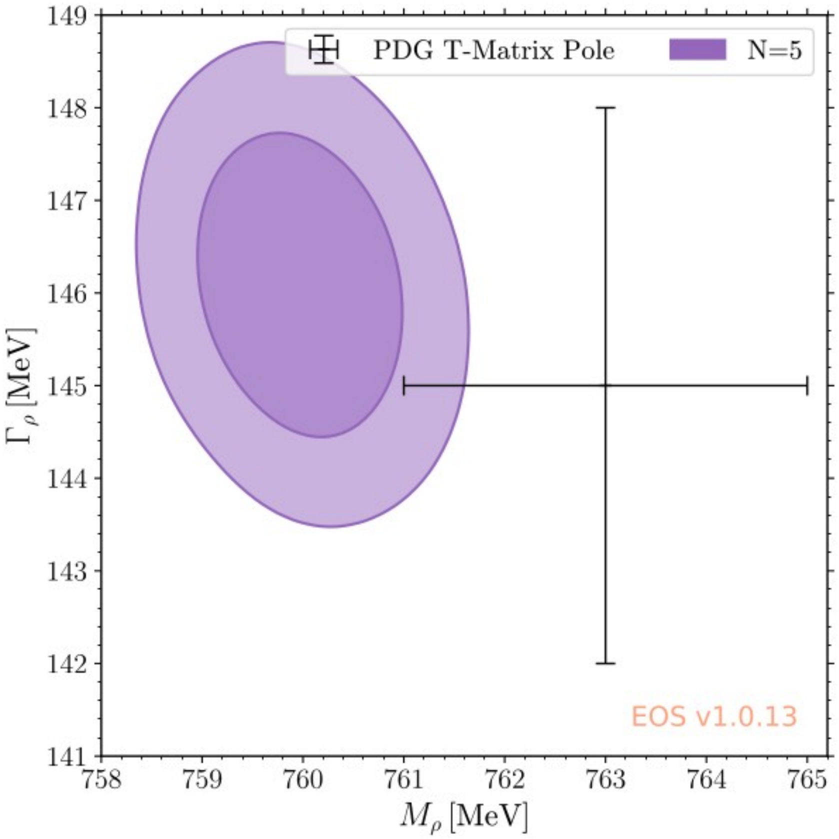


# $\rho$ pole parameters

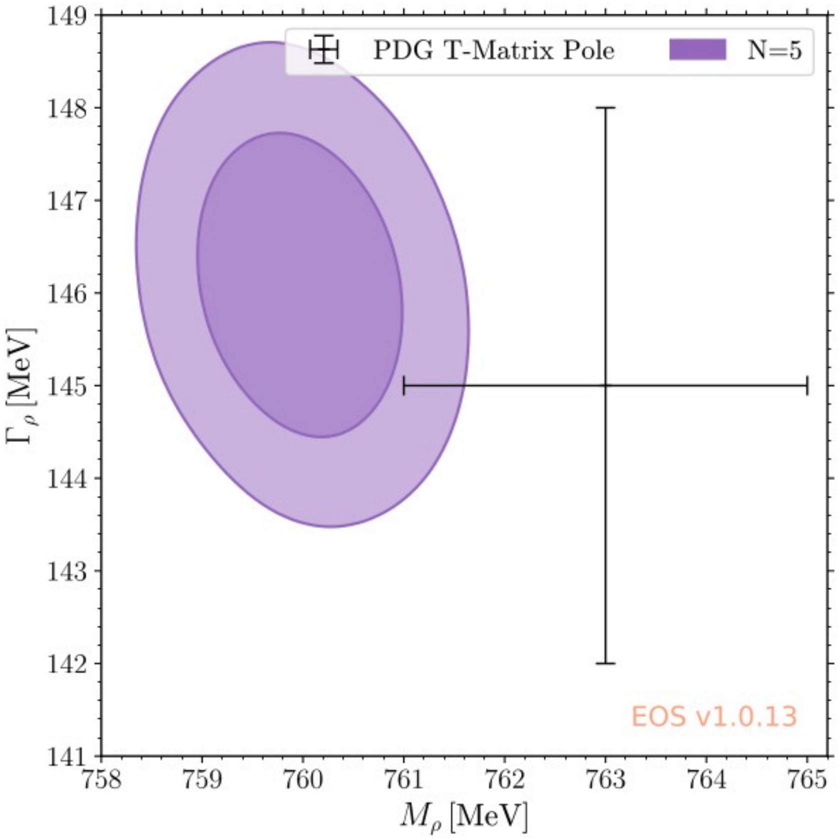
- Our N=5 fit gives

$$M_\rho = (760.0 \pm 0.6)\,\text{MeV}$$

$$\Gamma_\rho = (146.1 \pm 0.9)\,\text{MeV}$$

for $\rho$ pole location

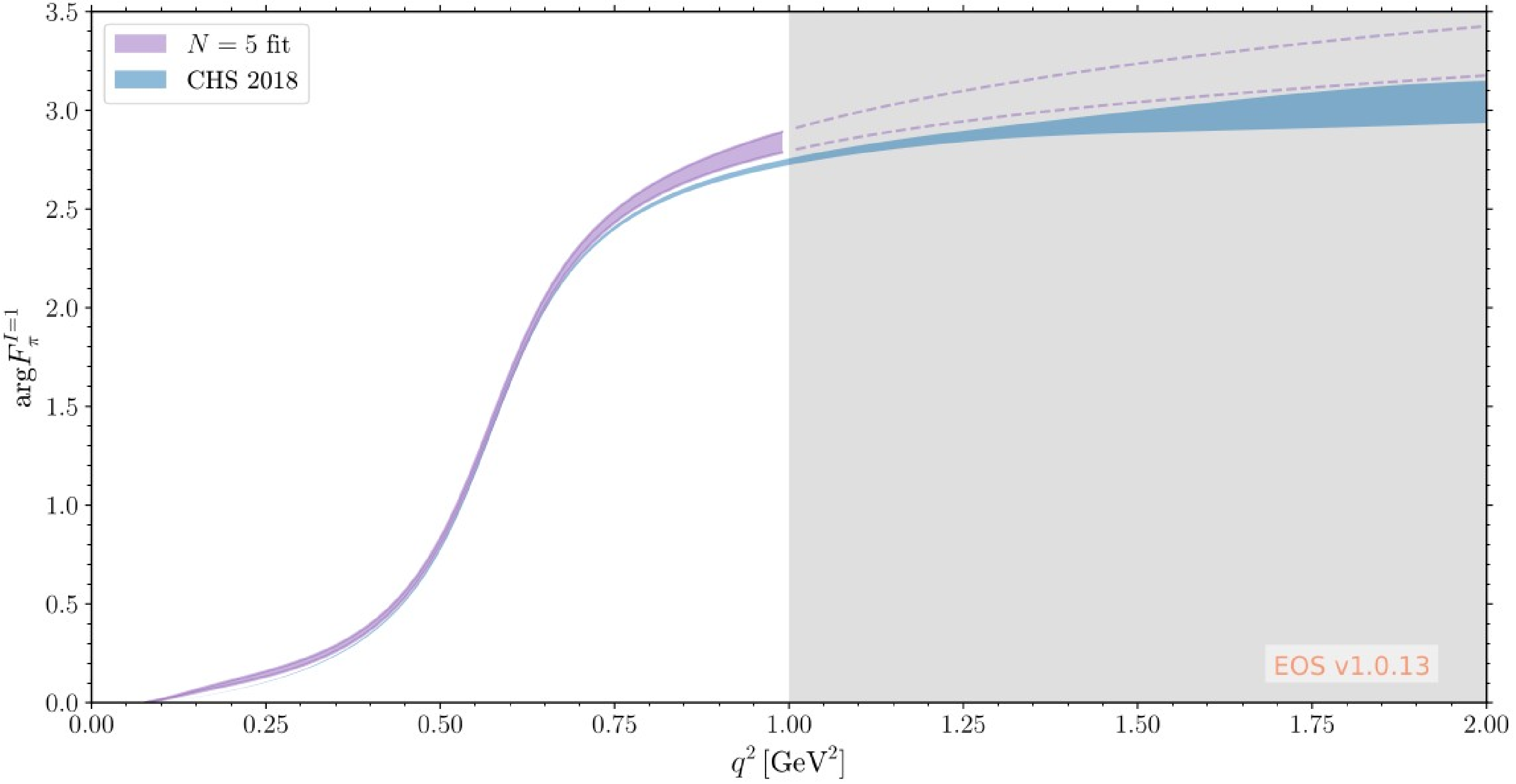
- Reasonable agreement with PDG which comes from other methods



# Alternative analyses

- Using analyticity, one can determine the magnitude if you know the phase on the branch cut up to infinity
- Extract the phase up to inelastic threshold, model the phase in the inelastic region

# Comparison to other work



# Future outlook and summary

# Going forward

- Now we have successful proof of concept, we are working on the $K \to \pi$ case
  - Allows a fit to $V_{us}$
- Ask me later about Cabibbo angle anomaly

# Summary

- We came up with a new way to parameterise form factors
  - Valid both above and below threshold, explicitly including resonance poles

# Summary

- We came up with a new way to parameterise form factors
- Allows to fit to data from all parts of phase space
  - But unlike other parameterisations, don't need phase data to infinity

# Summary

- We came up with a new way to parameterise form factors
- Allows to fit to data from all parts of phase space
- Proof of concept for pion form factor
  - Clear how to extend to e.g. $K \to \pi$, isospin breaking in pions, ...

# BACKUP

# Experts: why not Blaschke factors?

- For subthreshold poles, one can multiply by a Blaschke factor
  - $B(z; z_r) = \frac{z - z_r}{1 - z z_r^*}$
- Which removes a pole at $z = z_r$
- Above threshold, $|B(z; z_r)| = 1$ so dispersive bound simplifies better

# Why not Blaschke factors?

- Could we not write our form factor as
  - $F = \frac{W}{\phi} \frac{1}{B(z;z_r)} \frac{1}{B(z;z_r^*)} \sum b_i f_i$?
- Since this still has the pole at the $\rho$ ?

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  - $F = \frac{W}{\phi} \frac{1}{B(z;z_r)} \frac{1}{B(z;z_r^*)} \sum b_i f_i$?
- Since this still has the pole at the $\rho$ ?
- No! Now it has two zeros at $z = 1/z_r^{(*)}$, which are inside the unit circle
  - While in general some FFs are known not to have zeros on first Riemann sheet ❌

# Constraints on FF

- Want $F(0) = 1$
- Define $z_0 = z(q^2 = 0)$
  - So $1 = F(0) = \frac{W(z_0)}{\phi(z_0)} \sum_i b_i z_0^i$
- One condition on the $b_i$

# Constraints on FF

- We want $\mathrm{Im}\, F(q^2 \sim t_+) \sim (q^2 - t_+)^{3/2}$
- Note $z(t_+) = -1$, $z + 1 \propto (q^2 - t_+)^{1/2}$
- Expand f around -1:
  - $F(z \sim -1) \sim a + b(z-1) + c(z-1)^2 + d(z-1)^3 + \cdots$
  - $F(q^2 \sim t_+) \sim a + B(q^2 - t_+)^{1/2} + C(q^2 - t_+)^1 + D(q^2 - t_+)^{3/2}$
- Impose $0 = df/dz|_{z=-1} = b$ <= another condition

# Pion form factor data

- Data exists on the pion FF in several different kinematic regions
- From NA7 paper:

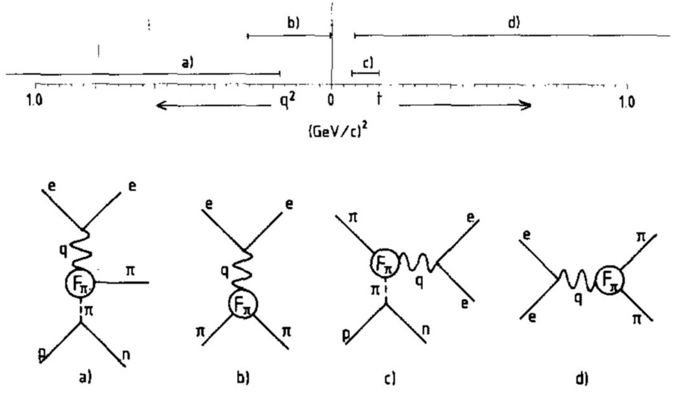
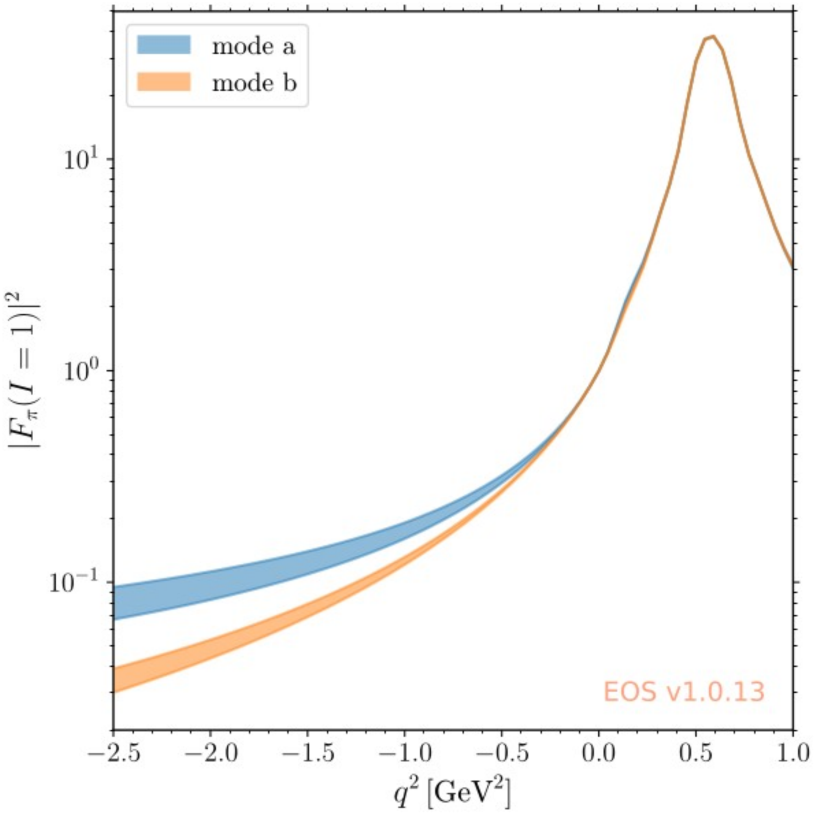
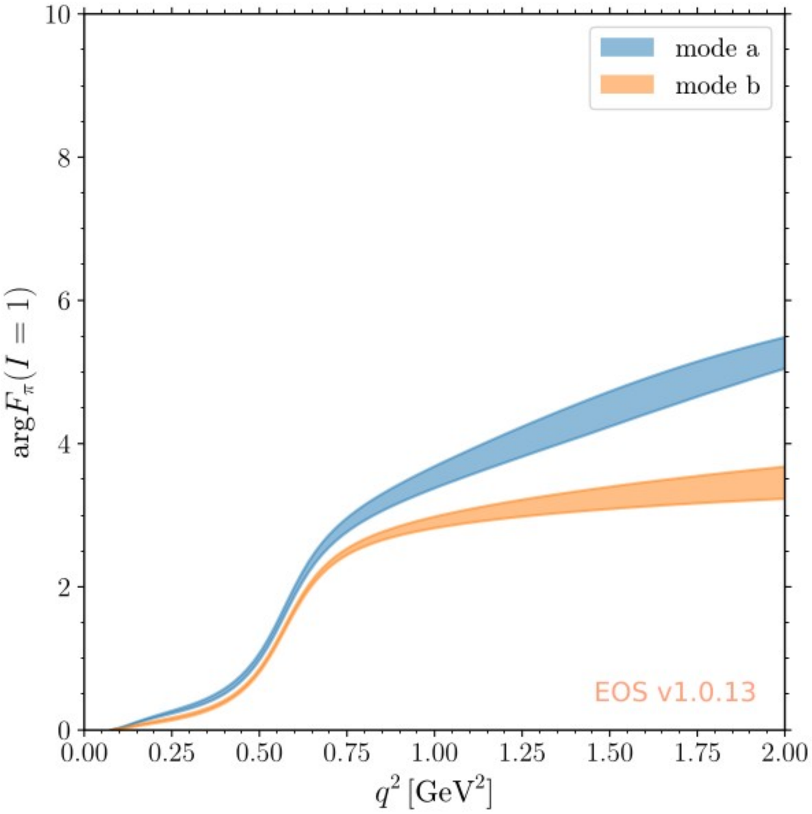



Fig. 1. Data on the squared modulus of $F_\pi$ for $|t| < 1(\text{GeV}/c)^2$ from the reactions: (a) electroproduction [1]; (b) direct $\pi$e scattering [2–4]; (c) inverse electroproduction [5]; and (d) $e^+e^-$ annihilation [6–9]. The horizontal bar (b) indicates the range of our experiment.

# Zeros on real axis



# Fitting semi-leptonic data

- $F = \frac{1}{\phi} \sum_i \alpha_i z^i$
- For semi-leptonic region, $z(q^2)$ is real and $|z| < 1$
  - E.g. for $B \to D$, can choose $t_0$ such that $|z| < 0.04$, for $B \to K$, $|z| < 0.3$
- The sum converges, and quickly